QUASI-COMPLETENESS OF SECOND-ORDER MODAL LOGIC S5 AND COMPLETENESS OF FIRST-ORDER S5[1]

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In the present article we frequently refer to our earlier article 'La correction de la logique modale du premier et second ordre S5' (*Logique et Analyse*, 1). We refer to this as 'CLM'.

This article contains six sections and thirty three paragraphs. The references will take the form 'CLM, IV' or 'CLM, 12', referring respectively to the fourth section and to the twelfth paragraph.

I:- *Quasi-semantic definitions for second-order logic*

0. The language of second-order modal logic contains all the propositions of the non-modal logic of second order; these are the propositions of the language L defined in CLM, 2 which do not contain modal symbols. If the second-order modal logic defined in CLM, IV is complete in the sense defined in CLM, 4 it follows that all the propositions valid in non-modal second-order logic will be derivable in second-order S5.

Now the set of derivable propositions in second-order S5 is clearly recursively enumerable. In particular the set of non-modal propositions is recursively enumerable. But from Gödel's incompleteness theorem it follows that the set of valid formulae of non-modal second-order logic is not recursively enumerable. We must conclude that second-order S5 (which we shall call S5.2[2]) cannot be complete.

This impossibility does not exist for first-order S5 and we shall prove the completeness of this logic.

All the same Henkin has shewn that non-modal second-order logic is complete in an extended sense which we may call 'quasi-complete'. We prove that S5.2 is quasi-complete in an analogous sense. In effect our exposition is no more than Henkin's theorem adapted for S5.

1. Let U be a universe composed of a set A of individuals and a set B of worlds and let a and b be the cardinal numbers of A and B respectively. In CLM, 1 we assumed, for each natural number n, a number c = 2exp (b(a exp n)) of n-place intensional predicates.

Assume, for each natural number n, a non-empty set Pn of n-place intensional predicates based on U. The sets A, B, P0, P1, P2, ... based on U constitute a quasi-universe Q based on U.

If for every natural number n, Pn contains all the n-place intensional predicates in U, Q will be a complete quasi-universe based on U. In such a case we say that all the intensional predicates in U are equally relative to Q.

2. We take a second-order modal language L defined as in CLM, 2. Consider a quasi-universe Q composed of the set A of individuals and B of worlds and sets of intensional predicates P0, P1, P2, ... We agree that the variables for individuals of the language L take as the values the individuals of the set A and that for each natural number n the variables for n-place predicates take as their values the intensional predicates in Pn.

If, in accordance with this convention, we are given a value to each of the variables of L we are given a value-system S relative to the quasi-universe Q.

3. We take a quasi-universe Q, a world M of this quasi-universe and a system of values S relative to this quasi-universe. We then define as follows the notions 'quasi-true for quasi-universe Q, the world M and value system S', and 'quasi-false for quasi-universe Q, world M and value system S'.

Let f be a proposition of language L.

If f is a variable p for 0-place predicates (i.e. a propositional variable), if P is the 0-place intensional predicate given as the value of p, f will be quasi-true or quasi-false for QMS according as P takes the value 'true' or 'false' when it receives M as argument.

If f is of the form bx1, where b is an n-place predicate variable ($n \neq 0$) and where x1,...,xn are individual variables, if B, X1,..., Xn are respectively the n-place intensional predicate and the individuals given as values of b, x1,..., xn, f will be quasi-true or quasi-false for QMS according as B takes the values 'true' or 'false' when it receives M, X1, ..., Xn as arguments in that order.

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[1]Translation of 'Quasi-adéquation de la logique modale de second ordre S5 et adéquation de la logique modale de premier ordre S5', *Logique et Analyse*, 2, 1959, 99–121.

[2]Bayart sometimes has S5.2 and sometimes S5,2. Probably a better terminology should be used.

If f has the form Np, where p is a proposition, f will be quasi-true for QMS if p is quasi-false for QMS, and quasi-false for QMS if p is quasi-true for QMS.

If f has the form Kpq, where p and q are propositions, f will be quasi-true for QMS if p and q are quasi-true for QMS, and quasi-false for QMS if not.

If f has the form Apq, where p and q are propositions, f will be quasi-true for QMS if p is quasi-true for QMS or if q is quasi-true for QMS, and quasi-false for QMS if not.

If f has the form Cpq, where p and q are propositions, f will be quasi-true for QMS if p is quasi-false for QMS or if q is quasi-true for QMS, and quasi-false for QMS if not.

If f has the form Epq, where p and q are propositions, f will be quasi-true for QMS if p and q are quasi-true for QMS or if p and q are quasi-false for QMS, and quasi-false for QMS if not.

If f has the form Pvp where p is a proposition and v a variable for individuals or predicates f will be quasi-true for QMS if for each system S′ relative to Q which gives to all the variables other than v the same values as S, p is quasi-true over QMS′. Otherwise f is quasi-false for QMS.

If f has the form Svp where p is a proposition and v a variable for individuals or predicates f will be quasi-true for QMS if there is a system S′ relative to Q which gives to all the variables other than v the same values as S, and according to which p is quasi-true over QMS′. Otherwise f is quasi-false for QMS.

If f has the form Lp, where p is a proposition, f will be quasi-true for QMS if for every world M′ of the quasi-universe Q, p is quasi-true for QM′S, and otherwise f will be quasi-false for QMS.

If f has the form Mp, where p is a proposition, f will be quasi-true for QMS if there is a world M′ of the quasi-universe Q such that p is quasi-true for QM′S, and otherwise f will be quasi-false for QMS.

4. We take a quasi-universe Q and a world M of this quasi-universe. We define for propositions of the language L the notions ‘quasi-valid in QM’ and ‘quasi-satisfiable in QM’.

Let f be a proposition of L.

The proposition f will be quasi-valid in QM if and only if, for each system S of values relative to Q, f is quasi-true for QMS.

The proposition f will be quasi-satisfiable in QM iff there is a system S of values relative to Q such that f is quasi-true for QMS.

Take a quasi-universe Q. We define for propositions of the language L the notions of ‘quasi-valid in Q’ and ‘quasi-satisfiable in Q’.

Let f be a proposition of L.

The proposition f will be quasi-valid in Q iff it is quasi-valid in every QM (for every world M).

The proposition f will be quasi-satisfiable in Q iff there is some world M such that f is quasi-satisfiable in QM.

We can express L in a deductive system D by being given axioms and rules of deduction. Assume a quasi-universe Q.

The deductive system D is quasi-sound for Q if one can only prove in D formulae which are quasi-valid in Q.

The deductive system D is quasi-complete for Q if one can prove in D all formulae which are quasi-valid in Q.

5. It is easy to check that S5.2 is not sound with respect to every quasi-universe. Consider for instance a quasi-universe which for 0-place intensional predicates contains only the predicate which takes the value false at every world. In S5,2 one can easily deduce the sequent I,Spp, where p is a propositional variable. But Spp is not satisfiable in the present quasi-universe. So, to develop the quasi-soundness of S5,2 we must invoke the notion of a ‘regular quasi-universe’ as follows.

In CLM,9 we gave a semantic definition of the value of an n-place parapredicate. We must now give the definition of the value of a proposition for a universe U and a value-system S. Let p be a proposition of L. The value of p for US is the 0-place intensional predicate which takes, for every world M of U, the value true or false according as p takes the value true or false for UMS.

We now give the following quasi-semantical definitions for a quasi-universe Q based on a value system S relative to Q.

The value of a proposition p for QS is the 0-place intensional predicate which, for any world M of U, takes the value true or false according as p is quasi-true or quasi-false.

The value of an n-place parapredicate Zx1...xn(p) for QS is the n-place intensional predicate which, for every world M of U, and every series of individuals A1,..., An takes the value true or false according as the proposition p takes the value quasi-true or quasi-false for MS′, where S is a value-system which assigns the individuals A1,..., An as values of the individual variables x1,..., xn respectively, and which gives all other variables the same values as S.

It is easy to see that the value of a proposition or of a parapredicate is not always an intensional predicate relative to Q. Thus, in the quasi-universe described above the propositional variable p can only take a single value, and in the given value-system the value of Np is not relative to Q.

A quasi-universe Q is regular if, for every proposition p of the language L, for every parapredicate Zx1...xn(p) constructed in the language L, and for every value system S relative to Q, the value of p and the value of Zx1...xn(p) is an intensional predicate relative to Q.

It is clear that regular quasi-universes exist, notably the complete quasi-universes. The present exposition will shew that there also exist regular incomplete quasi-universes.

6. We can now present the series of our quasi-semantical definitions:

A proposition is quasi-valid if and only if it is quasi-valid in all regular universes.

A proposition is quasi-satisfiable if and only if there is a regular universe in which it is quasi-satisfiable.

A deductive system D is quasi-sound all propositions derivable in D are quasi-valid.

A deductive system D is quasi-complete if one can prove in D all formulae which are quasi-valid.

II:- *Sematic properties of parapropositions*

7. In what follows we adapt the semantic theorems of CLM,III. Certain of the quasi-semantical theorems which follow hold for every quasi-universe, others only hold for regular quasi-universes. We will indicate each time which of these is the case.

8. *Theorem I.* Consider a quasi-universe Q, two worlds M and M′ of U and any value system relative to U. If p is a modalised proposition then p has the same value (quasi-true or quasi-false) for QMS and QM′S.

9. *Theorem II.* Let p be a proposition containing only v1,..., vn as free variables. Consider any quasi-universe Q, a world M of Q and two value systems S and S′ relative to U which do not differ in the values assigned to v1,..., vn. Then p takes the same value (quasi-true or quasi-false) for QMS and QMS′. In particular if p is a closed proposition then for any two value systems S and S′ relative to Q, p takes the same value for QMS and QMS′.

10. *Theorem III.* Let p be a proposition containing only v1,..., vn as free variables. Consider any quasi-universe Q based on a universe U, and two value systems S and S′ relative to U which do not differ in the values assigned to v1,..., vn. Then p takes the same value (see paragraph 5 above) for QS and QS′. In particular if p is a closed proposition then for any two value systems S and S′ relative to U, p takes the same value for QS and QS′.

We could have formulated a semantic analogue of theorem III in CLM, III.

11. *Theorem IV.* Let k be a parapredicate Zx1...xn(p) which contains only the variables v1,..., vn free. Take any quasi-universe Q, any world M of Q and any two value systems S and S′ relative to U which do not differ in the values given to the variables v1,..., vn. Then k takes the same value for QS and for QS′. In particular if k is a closed parapredicate then for each quasi-universe Q and for any two value systems S and S′ relative to Q, k takes the same value for QS and QS′.

The value of the proposition p in theorem III and that of the predicate k in theorem IV are the values relative to U and not necessarily values relative to Q.

12. *Theorem V.* For any quasi-universe Q, and world M of Q and any value system S relative to Q, if Zx1...xn(p)a1...an is a well-formed simple primary paraproposition the value (quasi-true or quasi-false) for QMS of the resultant p′ of this proposition is the same as the value of p for QMS′, where S′ is the value system relative to Q which gives the individual variables x1,...,xn the individuals A1,..., An respectively, these being the individuals that S assigns to the variables a1,..., an respectively, and which gives all other variables the same values as S does.

13. *Theorem VI.* For any quasi-universe Q, any world M of Q, and any value system S relative to Q, if Zx1...xn(p)a1...an is a well-formed simple primary paraproposition, and if P is the n-place intensional

predicate which is the value for QS of the parapredicate Zx1...xn(p), the value for QS of the resultant p′ of the given paraproposition will be quasi-true or quasi-false for QMS according as the predicate P takes the value true or false when it is given as arguments the world M and the individuals A1,..., An these last being in this order the values given by S to the variables a1,..., an.

The predicate P relative to U is not necessarily relative to Q.

14. *Theorem VII.* For any regular quasi-universe Q, any world M of Q, and any value system S relative to Q, if Zy(p)q is a well-formed propositional paraproposition, the value for QMS of the resultant p′ of this paraproposition is the same as the proposition p for QMS′ where S is the value system relative to Q which assigns the propositional variable y the 0-place predicate P such that p is the value of the proposition q for QS, and which gives all the other variables the same value as S.

(The analogous theorem VI of CLM,15 could have been stated as follows: For any universe U, any world M of U, and any value system S relative to U, if Zb(p)k is a well-formed propositional paraproposition where b is an n-place predicate variable and k is an n-place parapredicate the value for UMS of the final resultant p′ of this paraproposition is the same as the value of the proposition p for UMS′ where S′ is the value system which assigns to the propositional variable y the 0-place predicate P such that p is the value of the proposition q for QS, and which gives all the other variables the same value as S.)

15. *Theorem VIII.* For any regular quasi-universe Q, any world M of Q, and any value system S relative to Q, if Zb(p)k is a well-formed predicate paraproposition where b is an n-place predicate variable and k is an n-place parapredicate the value for QMS of the final resultant p′ of this paraproposition is the same as the value of the proposition p for QMS′ where S′ is the value system which assigns to the variable b the value which the parapredicate k takes for QS and which gives all the other variables the same values as S.

In theorems VII and VIII, from the fact that Q is a regular quasi-universe, the intensional predicate P is relative to Q, and so it is possible to use the value system S′ described in these theorems.

16. *Theorem IX.* Let p be a proposition. Let v be a variable. Let w be a variable of the same type as v which does not occur, either free or bound, in p.

Let q be the proposition obtained by substituting in p the variable w for the variable v wherever the latter occurs bound (q being identical with p if v is not bound in p.) Then, for any quasi-universe Q, any world M and any value system S relative to Q, p and q have the same value for QMS.

Proof by induction on the construction of p, distinguishing between cases where p has the form Pvj or Svj, and those where p has the form Puj or Suj, u being a variable distinct from v and w.

In CLM,III we could have formulated a sematic theory analogous to the present theorem IX, but such a theorem is not needed.

III:- *Quasi-soundness and quasi-completeness of S5, 2*

17. We say that a proposition p is derivable in S5,2 if the sequent I,p is derivable in S5,2.

We say that a sequent ä,I,ë is quasi-true for QMS if ä contains a proposition quasi-false for QMS or if ë contains a proposition quasi-true for QMS. Otherwise the sequent ä,I,ë is false for UMS.

One can then easily define quasi-validity and quasi-satisfaction for sequents.

We say that the proposition p represents the sequent ä,I,ë if p is a disjunction whose disjuncts, in order, are the negations of the propositions in ä followed by the propositions in ë. One can easily shew that ä,I,ë is derivable in S5,2 iff p is derivable in S5,2.

One can equally easily shew that ä,I,ë is quasi-true or quasi-false for QMS, iff p is quasi-true or quasi-false for QMS.

It follows that the quasi-soundness and quasi-completeness of S5,2 can be equally defined in terms of propositions or in terms of sequents.

18. *Theorem X.* If all propositions derivable in S5,2 are quasi-satisfiable in a quasi-universe, then all propositions derivable in S5,2 are quasi-valid in Q.

Proof from the fact that if a proposition p is derivable in S5,2 the proposition LPp is equally so. Pp designates here the universal closure of p.

19. *Theorem XI.* If S5,2 is quasi-sound for a quasi-universe Q, Q is a regular quasi-universe.

Proof: From the definitions of a quasi-sound system and a regular quasi-universe, and from the fact that all propositions of the form SbLPx1...PxnE.bx1...xn.q, where b is an n-place predicate variable, and where x1,..., xn are n distinct individual variables, and where q is a proposition not containing free b, and thus all propositions of the form SpLEpq, where p is a propositional variable, and where q is a proposition not containing free p, are derivable in S5,2.

20. *Theorem XII.* S5,2 is quasi-sound

The proof is analogous to the proof of the soundness of S5,2, given in CLM,IV. It must take account of the fact that quasi-soundness has been defined in paragraph 6 above in terms of regular quasi-universes.

The soundness proof for PI (see CLM,21) is based on the quasi-semantical theorems V, VII or VIII. Because the universes considered are regular it is possible to provide a value system S which gives to the variable v the value given by S to the argument a of the paraproposition Zv(p)a.

21. *Theorem XIII.* If p is a consistent proposition, i.e., if the sequent p,I is not derivable in S5,2, p is quasi-satisfiable.

Proof: Section IV of the present article will establish, for every consistent proposition p, a regular quasi-universe Q such that p is satisfiable in Q.

22. *Theorem XIV.* S5,2 is quasi-complete

proof: If p is quasi-valid, Np will be a proposition which is not quasi-satisfiable. By contraposition of theorem XIII we obtain that the sequent Np,I is derivable, from which it easily follows that the sequent I,p is derivable.

IV:- *Proof of theorem XIII*

23. In what follows we understand by 'proposition' a proposition of language L defined in CLM2 and by 'proposition or derivable sequent' we mean a proposition or sequent derivable in S5,2.

We use capital letters B, D, F etc., (i.e., letters other than N, K, A, C, E, P, L, M, Z and I) to designate propositions. These letters may be followed by one or two numerical indices.

The expressions B^0, D^0, F^0 etc., (i.e., letters other than N, K, A, C, E, P, L, M, Z and I) designate series or finite or infinite sets of propositions. These expressions may be followed by one or two numerical indices.

Use of these syntactical notations may be combined with the preceding syntactical notations.

If all the propositions of a set or series B^0 of propositions are elements of a set D^0 of propositions we say that the set or series B^0 is drawn from the set D^0.

24. A finite or infinite set B^0 of propositions is consistent if there is no finite series ä included in B^0 such that ä,I is derivable.

A finite or infinite series of propositions is consistent if it is included in a consistent set.

A proposition p is consistent with the set B^0 of propositions if the set B^0 + p is consistent.

It is easy to shew that if ä is a finite series of propositions included in a consistent set B^0, and if ä,I,p is derivable then p is consistent with B^0. A fortiori, if I,p is provable it is consistent with every consistent set.

25. Let y be a consistent proposition. We order the set of propositions of the form Mp in a series B0, B1, B2,... We order the set of propositions of the form Svp where v is any variable in a series D1, D2, D3,...

Consider the set of ordered pairs of natural numbers and order it diagonally as follows: 00, 01, 10, 11, 20,03, ... Assume the following series of propositions F0.0, F0.1, F1.0 ...

For each natural number n, Fn.0 is the proposition KMyCMpp where Mp = Bn.

For each pair of natural numbers n and m such that m ≠ 0, Fn.m is the proposition CSvpp′ where Svp = Dm and where p′ = Zv(p)a, a designating the first variable in alphabetical order of the same type as v which does not occur free in Svp nor in any proposition Fr.s where 'r.s' is an index which precedes 'n.m'.

We assume the following set of propositions G0.0, G0.1, G1.0 ... For each natural number n Gn0 is the proposition Mp where p = Fn.0.

For each pair of natural numbers n and m such that m ≠ 0, Gn.m is the proposition MK...Kp0...pm where p0,..., pm are respectively the propositions Fn.0,..., Fn.m.

26. Consider the set G^0 of propositions G0.0, G0.1, G1.0 ...

Lemma I. The set G^0 as defined above is consistent

Proof by reductio. Let ä be a finite series included in G^0 such that ä,I is derivable. Let Gn.m be the proposition of ä such that no other proposition of ä has an index of higher rank than n.m. let ä′ be the series composed of all the propositions Gr.s appearing or not in ä whose index is lower than n.m, and let Gn.m = p. It is clear that if ä,I is derivable then p,ä′,I is also.

We shew that this is impossible by induction on the rank of the index n.m.

Suppose n = m = 0. Then G0.0 is a proposition of the form MKMyCMpp and ä is empty. We then suppose that MKMyCMpp,I is provable. As we have KMyCMpp,I,MKMyCMpp we obtain by a cut that KMyCMpp,I is derivable. As we have My,CMpp,I,KMyCMpp we obtain by a cut that My,CMpp,I is derivable. Since My is modalised we have that MCMpp,My,I is derivable.

But I,MCMpp is derivable as follows:

	Mp,p,I,p
	________________
Mp,I,p,Mp	p,I,CMpp
________________	________________
I,CMpp,Mp	p,I,MCMpp
________________	________________
I,MCMpp,Mp	Mp,I,MCMpp
____________________________________	
I,MCMpp	

Hence by cut with MCMpp,My,I we obtain that My,I is derivable, contrary to the hypothesis according to which it is a consistent proposition.

Suppose n ≠ 0 and m = 0. Gn.m then has the form MKMyCMpp but ä′ is no longer empty.

Suppose then that MKMyCMpp,ä′,I is derivable. We deduce successively that the following sequents are derivable:

KMyCMpp,ä′,I	
My,CMpp,ä′,I	
MCMpp,My,ä′,I	(since all the propositions in ä are modalised.)
My,ä′,I	(since I,MCMpp is derivable.)

But ä contains the proposition G0.0 which has the form MKMyCMqq. Call this proposition ‘g’. Now we have the following proof:

My, CMqq,I,My

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KMyCMqq,I,My

____

MKMyCMqq,I,My

I.e., that g,I,My is derivable, whence by a cut with My,ä′,I we obtain g,ä′,I.

But g is a proposition of ä′. Thus we have ä′,I contrary to the induction hypothesis.

Suppose n is any number and m ≠ 0. Gn.m has then the form MK...Kp0...pm where pm has the form CSvqq′ where q′ is Zv(q)a. We then suppose that MK...Kp0...pm,ä′,I is derivable. As Gn.m has an index of higher rank than all the other propositions of ä′, and as pm is the proposition Fn.m of which the index is of greater rank than all the other propositions which enter into the composition of Gn.m or of a proposition of ä′, we have that the variable a does not occur free or bound except in q′, i.e. in Zv(q)a.

Hence if MK...Kp0...pm,ä′,I is derivable, the following sequents are also

K...Kp0...pm,ä′,I

K...Kp0...pm−1,pm,ä′,I or, what amounts to the same

K...Kp0...pm−1,CSvqq′,ä′,I

SaSvqq′,K...Kp0...pm−1,ä′,I (in virtue of what has been said about the variable a.)

But I,SaCSvqq′ is derivable as follows:

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                                Svq,q,I,q
                                _________
Svq,I,q,Svq                     q,I,CSvqq        (1)
_____________                   _________
     I,CSvqq′,Svq               q,ISaCSvqq′
(3) ____________                _________        (2)
     I,SaCsvqq′,Svq             Svq,I,SaCSvqq′
________________________________________________
               I,SaCSvqq′
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To enable verification of the legitimacy of this proof it is pointful to make the following remarks

(1) q′ = Zv(q)a where a has no free or bound occurrences in q. It follows from this that q = Za(q′)v and that CSvqq = Za(CSvqq′)v
(2) The variable a does not occur free in SaCSvqq′.
(3) CSvqq′ = Za(CSvqq′)a

From SaCSvqq′, K...Kp0...pm-1,ä′,I and from I,SaCSvqq′ we obtain by a cut K...Kp0...pm-1,ä′,I. Noting that all the propositions of ä′ are modalised we obtain MK...Kp0...pm-1,ä′,I. But MK...Kp0...pm-1,ä′ is a proposition of ä′. Hence we obtain ä′,I, contrary to induction hypothesis. This completes the proof of the lemma.

27. Consider the set of all modalised propositions and order this in a series H1, H2, H3, ... We assume the following series of sets of propositions $H^0$0, $H^0$1, $H^0$2, ...

$H^0$0 = G^0.

H^0n+1 = H^0n if the proposition Hn+1 is inconsistent with H^0n and otherwise H^0n+1 = H^0n + Hn+1

We see immediately by induction on n, and noting that G^0 is consistent, that for every n, H^0n is consistent.

Let H^0 be the union of $H^0$0, $H^0$1, $H^0$2,...

Lemma II. H^0 is consistent.

Proof by reductio. Let ä be a series included in H^0 such that ä,I is derivable. Let Hn be the proposition with the highest index in ä. It is clear that all the propositions of ä appear in H^0n. Then H^0n will be inconsistent, contrary to construction.

Lemma III. If p is a modalised proposition then if p is consistent with H^0 then p is an element of H^0.

Proof: Let the index of p in the series H1, H2, H3 be n. If p is consistent with H^0 then it is consistent with H^0n−1. From this we have by construction that H^0n = H^0n−1 + p. So p is an element of H^0.

28. Assume the series $F^0$0, $F^0$1, $F^0$2 containing respectively the propositions F0.0, F0.1, F0.2, ..., F1.0, F1.1, F1.2, ..., F2.0, F2.1, F2.2, ...,

Assume the series $Q^0$0, $Q^0$1, $Q^0$2, ... defined as follows: $Q^0$0 = H^0 + $F^0$0; $Q^0$1 = H^0 + $F^0$1; $Q^0$2 = H^0 + $F^0$2, ...

Lemma IV. The sets $Q^0 0$, $Q^0 1$, $Q^0 2$, ... are consistent.

Proof by reductio. Consider some series $Q^0 n$. Let ä be a series included in $Q^0 n$ such that ä,I is derivable. Let ä′ be the series composed of those elements of ä which are elements of $F^0 n$ and let ä″ be that which remains in the series ä when all the elements of ä′ are removed. Let ä‴ be the series Fn.0, ..., Fn.m where m is the highest number occurring in the second index of a proposition in ä′. It is clear that if ä,I is derivable then ä‴,ä″,I [o‴] is equally. Consider the proposition K...Kp0...pm where p0,..., pm are respectively the propositions Fn.0, ..., Fn.m. We would have that K...Kp0...pm,ä″,I is derivable. Taking account of the fact that all the propositions of ä are elements of H^0 and thus are modalised propositions we would have that MK...Kp0...pm,ä″,I is derivable. But MK...Kp0...pm = Gn.m and Gn.m like all the propositions of ä is an element of H0. It follows that H^0 would be inconsistent, contrary to lemma II.

It is clear that identical reasoning holds equally for the case where ä contains only the proposition Fn.0.

29. Consider the set of all propositions and order them in a series R1, R2, R3 ... defined as follows:
For each number n R0n.1 = Q0n. For each number m+1 $R^0 n.m+1 = R^0 n.m$ if Rm+1 is inconsistent with $R^0 n.m$ and otherwise $R^0 n.m+1 = R^0 n.m + Rm+1$. we see immediately by induction on m, and considering that $Q^0 n$ is consistent, that for each m $R^0 n.m$ is consistent.

Consider the sets $R^0 0$, $R^0 1$, $R^0 2$ ... which are respectively the unions of the sets $R^0 0.1$, $R^0 0.1$, $R^0 0.2$ ...$R^0 1.0$, $R^0 1.1$, R01.2, ... $R^0 1.2$ ... $R^0 2.0$, $R^0 2.1$, $R^0 2.2$, ...

Lemma V. The sets $R^0 0$, $R^0 1$, $R^0 2$ ... are consistent.

Proof by reductio. Let ä be a series included in $R^0 n$ such that ä,I is derivable. Let Rm be the proposition of ä whose index m is the highest. It is clear that all the propositions of ä appear in $R^0 n.m$. Hence $R^0 n.m$ is inconsistent, contrary to construction.

Lemma VI. Let p be a proposition. If p is consistent with $R^0 n$ p is an element of $R^0 n$.

Proof: Let the index of p in the series R1, R2, R3 be m. If p is consistent with $R^0 n$ it is consistent with $R^0 n.m+1$. from this we have, by definition, that $R^0 n.m = R^0 nm-1 + p$. So p is an element of $R^0 n$.

Lemma VII. If p is a modalised proposition and if p appears in a set $R^0 n$ then, for all m, it appears in $R^0 m$.

Proof: Let i be the index of p in the series R1, R2, R3, ... If p belongs to $R^0 n$ then p is consistent with $R^0 ni-1$. But $R^0 ni-1$ contains H^0. So p is an element of H^0. From this, in virtue of the manner of definition of the set $R^0 0$, $R^0 1$, $R^0 2$ ... p is an element of each of these sets.

30. Assume a universe U containing a denumerably infinite set of individuals and a denumerably infinite set of worlds.

We establish a 1-1 correspondence between individual variables and the individuals of U.

We establish a 1-1 correspondence between the sets $R^0 0$, $R^0 1$, $R^0 2$ ... and the worlds of U. Consider the set of intensional predicates which are given by U. For each natural number n we establish a correspondence between n-place predicate variables and certain n-place intensional predicates such that to each variable corresponds a single predicate, though several variables may correspond to the same predicate.

If p is a propositional variable we let correspond to p the 0-place intensional predicate P which takes the value 'true' for the worlds corresponding to the sets Rn which contain p, and the value 'false' for the other worlds.

If b is an n-place predicate variable ($n \neq 0$) we let correspond to b the n-place intensional predicate B which, when given as arguments a world M and the individuals X1, ..., Xn (not necessarily distinct), takes the value 'true' or 'false' according as the proposition bx1...xn is contained or not in the set Rm, the set Rm being that which corresponds to the world M and the variables x1,..., xn being those which correspond to the individuals X1, ..., Xn respectively.

Consider the set of intensional predicates of U, which we have made correspond with the variables of L. This set of predicates constitutes, with the set of individuals and the set of worlds of U, a quasi-universe Q included in U. Further, the system of correspondences established constitutes a value-system S, relative to Q. It is clear that the quasi-universe Q permits the establishing of other value system than S.

31. *Lemma VIII.* Let Q be a quasi-universe and S the value system relative to Q corresponding with the set R^0m. Let p be a proposition. Then p is quasi-true or quasi-false for QMS according as p occurs or not in R^0m.

Proof by induction on the construction of p. (v. remarks at the end of the present paragraph.)

If p is an elementary proposition the lemma follows from the correspondences established between the variables of L and the quasi-universe Q.

If p has the form Ng and if Ng is in R^0m then g is not in R^0m, for otherwise R^0m would be inconsistent. So g is quasi-false for QMS and Ng is quasi-true for QMS.

If p, i.e. Ng, does not appear in R^0m, then g appears in R^0m, for if not it would follow that Ng and g are both inconsistent with R^0m. We would then have the derivable sequents Ng,ä,I and g,ä′,I, where ä and ä′ are sequents taken from R^0m. Let ä″ = ä + ä′. We then have Ng,ä″,I and g,ä″,I, and easily obtain ä″,I,g. By a cut with g,ä″,I we obtain ä″,I, and therefore that R^0m is inconsistent. If g is in R^0m, g is quasi-true for QMS, and so Ng is quasi-false.

If p has the form Kgj and p appears in R^0m, g and j appear in R^0m. For Kgj,I,g and Kgj,I,j are derivable. So g and j are consistent with R^0m, and from this are clearly in R^0m. So g and h are quasi-true for QMS, and so Kgj is quasi-true for QMS.

If p, i.e. Kgj does not appear in R^0m. g and j cannot both appear, for otherwise, since the sequent g.j,I,Kgj is derivable, Kgj would be in R^0m. One of the two propositions g and j will not be in R^0m, and this one will be quasi-false for QMS. So Kgj is quasi-false for QMS.

If p has the form Agj and p appears in R^0m, one of the propositions g and j will appear in R^0m, for otherwise Ng and Nj will appear, and by Ng,Nj,Agj,I, R^0m will be inconsistent. Whichever proposition g or j appears in R^0m will be quasi-true, and so Agj will be quasi-true for QMS.

If p, i.e., Agj does not appear in R^0m, then neither g nor j appear in R^0m. For otherwise, since g,I,Agj and j,I,Agj are derivable Agj will appear in R^0m. So g and j are quasi-false for QMS, and from this Agj is quasi-false for QMS.

If p has the form Cgj and p appears in R^0m, j will appear in R^0m or g will not be in R^0m, for otherwise g and Nj will appear, and since Nj,g,Cgj,I is derivable, R^0m will be inconsistent. If j appears in R^0m then j will be quasi-true for QMS, and if g does not appear in R^0m then g will be false for QMS, and in either case Cgj will be quasi-true for QMS.

If p, i.e., Cgj does not appear in R^0m, then j will not appear in R^0m and g will appear in R^0m. For otherwise, j or Ng will be in R^0m, and since j,I,Cgj and Ng,I,Cgj are derivable Cgj will appear in R^0m. So j is quasi-false for QMS and g is quasi-true for QMS, and from this Cgj is quasi-false for QMS.

If p has the form Egj and p appears in R^0m, g and j will both be in R^0m or neither g nor j will be in R^0m, For if one of these propositions is in R^0m and the other is not, one will have, for instance, that g and Nj are in R^0m. But Nj,g,Egj,I is derivable. It follows that g and j are both quasi-true for QMS or that g and j are both quasi-false for QMS, and so Egj is quasi-true.

If p, i.e., Egj does not appear in R^0m, then one of the propositions g and j will appear in R^0m and the other not. For, if both propositions appear then one notes that g,j,I,Egj is derivable, and if neither g nor j is in R^0m then Ng and Nj are in R^0m, and Ng,Nj,I,Egj is derivable. So one of the two propositions must be quasi-true for QMS and one quasi-false for QMS, and from this Egj is quasi-false for QMS.

If p has the form Pvg and if p occurs in R^0m, then for every value system S′ which gives all variables other than v the same value as S, g is quasi-true for QMS′. For let X be the entity (individual or predicate) which S′ makes correspond with the variable v and let a be the variable, of the same type as v, which S makes correspond with X. Two hypotheses arise according as Zv(g)a is a well-formed paraproposition or not. If Zv(g)a is well-formed let j be its resultant. Then, since Pvg,I,j is derivable, j, appears in R^0m and is thus quasi-true for QMS. But in virtue of theorems V, VII or VIII, j has, for QMS, the value which g has for QMS′. Thus g is quasi-true for QMS′.

If Zv(g)a is not well-formed it will be because v occurs free in g in the scope of a quantifier Pa or Sa. let g′ be the proposition obtained by replacing in g the variable a everywhere it occurs bound by a variable c of the same type which does not occur in Pvg, hence not in g, free or bound. Pvg,I,Pvg′ is derivable and hence Pvg′ is an element of R^0m. Further Zv(g′)a is well-formed and hence j′ is an element of R^0m and so quasi-true for QMS. It follows, in virtue of theorem IX, that g and g′ have the same value for QMS′. Thus g is quasi-true for QMS′. Then for all value systems S′ which give to all variables other than v the same value as S′, g is quasi-true for QMS′. Thus Pvg is quasi-true for QMS.

If p, i.e. Pvg, does not appear in R^0m there is a value system S′ which gives to all variables other than v the same values as S, such that g is quasi-false for QMS′. For, if Pvg does not appear in R^0m, NPvg appears in R^0m and as NPvg,I,SvNg is derivable, SvNg appears in R^0m. But R^0m contains a proposition of the form CSvNgNg′ where g′ is Zv(g)a, this paraproposition being well-formed. It follows that Ng′

appears in R^0m since SvNg,CSvNgNg′,I,Ng′ is derivable. So Ng′ is quasi-true for QMS and g′ is quasi-false for QMS. Let S′ be the value system which gives v the same value as S gives to a and to all variables other than v the same value as S. We have that g has the same value for QMS′ as g′ has for QMS. Thus g is quasi-false for QMS. [QMS′ ?] It follows that Pvg is quasi-false for QMS.

If p has the form Svg and if p occurs in R^0m, then there is a value system S′ which gives all variables other than v the same value as S, and g is quasi-true for QMS′. (We leave the proof to the reader who can adapt the proof given above for the case where p has the form Svg and does not appear in R^0m.) It follows that Svg is quasi-true for QMS.

If p, i.e. Svg, does not appear in R^0m, then for every value system S′ which gives to all variables other than v the same values as S, such that g is quasi-false for QMS′. It follows that Pvg is quasi-false for QMS. (We leave the proof to the reader who can use the proof given above for the case where p has the form Pvg and appears in R^0m.)

If p has the form Lg and if p appears in R^0m since Lg,I,g is derivable g is in R^0m and g is quasi-true for QMS.

Further, in virtue of lemma VII, for any number m′, Lg appears in $R^0m′$. It follows that for every world M′ g is quasi-true for QMS′, and from this that Lg is quasi-true for QMS.

If p, i.e. Lg, does not appear in R^0m, NLg appears in R^0m, and as NLg,I,MNg is derivable MNg appears in R^0m. Further for every number m′, MNg appears in $R^0m′$. Suppose that the proposition MNg is the proposition Bm′ (v. paragraph 25.) Then the proposition KMyCMNgNg,MNg,I,Ng is derivable, Ng is an element of $R^0m′$. It follows that if M′ is the world corresponding to $R^0m′$, g is quasi-false for QM′S and hence g is quasi-false for QM′S and hence Lg is quasi-false for QMS.

If p has the form Mg and if p appears in R^0m there is a world M′ such that g is quasi-true for QM′S. (We leave the proof to the reader, who can adapt the proof given above for the case where p has the form Lg and does not appear in R^0m.) It follows that Mg is quasi-true for QMS.

If p, i.e. Mg, does not appear in R^0m, then NMg appears in R^0m, and as NMg,I,LNg is derivable LNg will be in $R^0m′$ for every number m′, and from this, for every world M′, g is quasi-false for QM′S, It follows that Mg is quasi-false for QMS.

Remark: The proof cannot strictly be said to be by induction on the construction of p, but by induction on propositions with an identical structure. Two propositions are said to have the same structure if each can be obtained from the other by substitution of free or bound variables. Then, where p has the form Pvg and p is in R^0m we can assume that the lemma has been proved, not only for g, but also for Zv(g)a. Note also that, for instance, where p has the form Pvg and is not in P^0m we can suppose that the lemma has been proved, not only for g′ (g′ = Zv(g)a) but also for Ng′. This is clearly legitimate because we have already proved that if the lemma holds for g′ it holds for Ng′.

32. *Lemma IX*. The proposition y is quasi-satisfiable in the quasi-universe Q.

Proof: Suppose that My is the proposition Bm (see paragraph 25.) Then KMyCMyy is the proposition Fm.0 and it is in R^0m. Now, since we have KMyCMyy,I,y it follows that y is in R^0m and thus is quasi-true for QMS, where M is the world corresponding to R^0m.

33. *Lemma X*. The quasi-universe Q is regular.

Proof: For any number m all theorems are in R^0m. So all theorems are satisfiable in Q. By theorems X and XI Q is regular.

With the proof of theorem XIII we have established that if y is a consistent proposition there is a regular quasi-universe Q in which y is quasi-satisfiable.

V:- *Completeness of S5,1*

34. Recall that we are given the following: (1) the language S5,1, defined in CLM,23; (2) the semantic definitions of CLM, 3 and 4, which, as observed in CLM,24, are applicable to the language S5,1; (3) theorems I, II and IV of CLM, adapted, as has been said in CLM,26, to the language S5,1; (4) a semantic (not quasi-semantic) theorem analogous to theorem IX of the present article; for S5,1 the variables v and w of this theorem are individual variables; (5) the deductive system S5,1 defined in CLM,27.

35. We can make sets and series of propositions of the language S5,1 analogous to the sets and series defined in paragraphs 25-29 of the present article. Lemmas I-VIII can be proved as in those paragraphs.

36. We assume a universe U, and establish the correspondences described in paragraph 30 of the present article. We no longer need the quasi-universe Q containing just those intensional predicates which correspond with predicate variables. Lemma VIII can be read as follows:

Lemma VIII. Let U be a universe and S a value system relative to U. Let M be a world in U corresponding to the set R^0m. Let p be a proposition. Then p is true or false for UMS according as p is or is not in R^0m.

The proof is as in paragraph 31. We have the truth or falsity of p rather than quasi-truth or falsity because we don't have second-order quantifiers in the language S5,1. So if one looks at the series of quasi-semantical definitions of 'quasi-true' and 'quasi-false' given in paragraph 3 of the present article, one can see that they are equivalent to the notions 'true' or 'false'. In other words, for the language S5,2, the definitions of 'quasi-true' and 'quasi-false' are no different from those for 'true' and 'false' except where v is a propositional or predicate variable. The difference in those cases arises because one doesn't consider all the intensional predicates in U, but only those which occur in the quasi-universe Q based on U. It follows that for the language S5,2 a proposition containing second-order quantifiers might be quasi-true or quasi-false for QMS without being true or false for UMS and vice versa. The absence of second-order quantifiers in S5,1, makes this difference disappear.

It follows that we can proceed as follows: Apply the quasi-semantical definitions of paragraph 3 (not those of paragraph 4) to the language S5,1. In Lemma IV choose the quasi-universe Q, and not the Universe U. Prove, as in paragraph 31, that p is or is not quasi-true or quasi-false for QMS according as p is or is not in R^0m. We can claim that p is true or false for UMS according as p is or is not in R^0m, which is essentially lemma VIII relative to S5,1, as formulated above.

37. We can prove, as in paragraph 32,

Lemma IX. y is satisfiable in a Universe U.

Lemma X falls out of the collection of lemmas I-IX proved for S5,1

Theorem XV. If p is a consistent proposition in S5,1 then p is satisfiable.

From this one can conclude

Theorem XVI. S5,1 is complete.

38. It has been possible to adapt the Henkin proof method to S5,2 and S5,1. One might have considered adapting the Gödel proof method to S5,1. But one encounters a difficulty from the fact that the Gödel method rests on the technique of prenex formulae, and this technique is unavailable in modal logic.