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Approximate Truth and Nonstandard Analysis

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Abstract. In [9], Henson introduced a notion of approximate truth for normed structures. He showed that for nonstandard hulls of normed spaces truth and approximate truth were equivalent for the positive bounded formulas, i.e. for formulas constructed from the atomic ones allowing finite conjunction, disjunction and bounded existential and universal quantification.

We generalize the notion of approximate truth to formulas allowing infinite conjunction, negation and bounded quantification. We show that for this very expressive language Henson's result still holds. Finally, we give a brief description of the main properties of this notion of approximate truth.

1. Introduction

One of the most successful and widely applied techniques in analysis is the notion of *proof by approximation*:

> *To prove that a statement in mathematics is true it is often showed that various* approximations *to the statement hold, and then special properties of the structure in question are cited to conclude that the statement in question is exactly true.*

Examples of these proofs by approximation abound, from compactness arguments in topology, weak compactness arguments in functional analysis and probability, to existence of solutions of differential equations. Although proof by approximation is constantly used, it is only recently that a systematic treatment of it from a logic point of view has been started.

In 1976 Henson in [9] introduced a logic of positive bounded formulas in Banach spaces to study the relationship between a Banach space E and its nonstandard hull $H(E)$. This logic L_{PB} is based on a first order language L containing a binary function symbol $+$, unary predicate symbols P and Q to be interpreted as the closed unit ball and the closure of its complement and for every rational number r a unary function symbol f_r to be interpreted as the operation of scalar multiplication by r. L_{PB} is closed under finite conjunction, disjunction and bounded quantification of the form $(\exists x)(P(x) \wedge \ldots)$ or $(\forall x)(P(x) \Rightarrow \ldots)$.

For any formula ϕ in L_{PB} and for every natural number n it is possible to define in a purely syntactical way a formula ϕ_n in L_{PB}, called the *n-approximation* of ϕ. Intuitively speaking ϕ_n is the formula that results from weakening the predicates that appear in ϕ in such a way that as n tends to ∞, ϕ_n approaches ϕ. From this notion of approximation follows the definition of *approximate truth*: a formula ϕ is approximately true in a Banach space E (denoted by $E \models_{AP} \phi$) iff for every integer n, $E \models \phi_n$.

This definition is the starting point of the model theory of Banach spaces that has been developed on a series of papers by Henson, Heinrich and Iovino (see in [10, 7, 8, 6, 11]). The logic L_{PB} has been extended to have a language of n-ary function symbols (to be interpreted as uniformly continuous functions from E^n to E) and n-ary real valued relation symbols. The notions of approximate formula and approximate truth are generalized naturally, and nice theorems (in particular a compactness theorem, Löwenheim–Skolem theorems, etc.) were obtained. Furthermore, the model theory of approximate truth for those structures had recently been extended, obtaining stability results (see for example [12]).

In particular, it is easy to remark that in any model E and for any formula $\phi \in L_{PB}$, if $E \models \phi$ then $E \models_{AP} \phi$. Going the other direction, Henson in [9] obtained the following approximation principle for nonstandard hulls:

Theorem 1: Henson's Approximation Principle

> *For any nonstandard hull E of a Banach space, and for any formula ϕ in L_{PB}, $(E \models_{AP} \phi) \Leftrightarrow (E \models \phi)$.*

Since usually it is easier to check that a formula is approximately true that to verify that it is true, those results show that structures obtained from nonstandard analysis are ideally suited to perform proof by approximations. This idea has been exploited in [2, 3]. The central concept of approximation in those papers is not the approximate formulas, but the notion of neoforcing. In this framework, Fajardo and Keisler were able to extend Henson's Approximation Principle to a large collection of formulas.

The aim of this paper is to show that this approximation principle for approximate truth holds in fact for a very expressive logic that contains L_{PB}.

To achieve this, we define the general logic L_A and the semantics associated with it in Section 2. This logic is closed under negations, countable conjunctions and bounded existential quantification. We also give examples of the expressive power of this logic.

In Section 3 we introduce the notion of approximate truth for this logic and give some examples of the difference between truth and approximate truth. This notion is an extension of Henson's original definition to L_A.

In Section 4 we discuss some elementary properties of our definition and we define the *rich models* for a collection Δ of formulas in L_A. A model $\mathcal{A}$ is Δ-rich if the following approximation principle holds:

> *For every formula ϕ in Δ, $\mathcal{A} \models_{AP} \phi$ if and only if $\mathcal{A} \models \phi$.*

In this section we also give an easy test to verify if a model is Δ-rich, and we use it to show that the nonstandard hulls of metric spaces are rich models.

Finally, in Section 5 we give some concluding remarks.

2. Definition of the logic L_A

Our intention is to define a logic expressive enough to capture most properties of metric spaces that arise in Analysis. Most of those properties refer to maps between the structure in question and another metric space. Examples of such maps are the metric of a metric space, the expected value of a random variable, etc. In order to accomodate such functions we will need to deal with a logic that accepts multisorted predicate symbols. The full description of such a language is as follows.

2.1. The signature Φ

We will need to distinguish two different types of sorts:

- Fix a collection T of metric spaces. The elements of the collection T are going to be called the *fixed sorts* of the signature.
- A *true sort*. Intuitively, the true sort is going to be the metric space associated with every particular model.

A *signature* $(\mathcal{F}, \mathcal{P}, \mathcal{AP}, \mathcal{K})$ is defined as follows:

- $\mathcal{F}$ is a collection of symbols of functions such that each element $f \in \mathcal{F}$ has a corresponding arity $a_f < \omega$. The sort of f could be the true sort, or a fixed sort (M_f, ρ_f) in T. Intuitively, those symbols are going to be interpreted as maps from the model to itself, or from the model to the sort metric space (M_f, ρ_f).
- $\mathcal{P}$ is a collection of predicate symbols such that each element $C \in \mathcal{P}$ has a corresponding arity $a_C < \omega$. The sort of C could be the true sort, or a fixed sort (M_C, ρ_C). If the sort of the predicate is not the true sort, then its arity a_C should be 1. Intuitively, those symbols are

going to be interpreted as closed relations in a cartesian product of the model, or as closed relations in (M_C, ρ_C).

- $\mathcal{AP}$ is a collection of predicate symbols *disjoint* from P, such that for every C in $\mathcal{P}$, for every integer n, C_n is a predicate in $\mathcal{AP}$ with the same arity and sort space that C. C_n is called the n-approximate predicate of C and it represents a metric deformation of the predicate C.

- $\mathcal{K}$ is a collection of predicate symbols such that each element $K \in \mathcal{K}$ has a corresponding arity $a_K < \omega$ and its sort is the true sort. The elements of $\mathcal{K}$ are called the *bounding* predicates of Φ and are going to be interpreted as the sets bounding the quantifiers.

Let us see an example of a signature.

Example 1: Typical signatures

Suppose that the structures that we want to study are the Banach spaces with the following basic operations:

(a) Sum of two vectors, and multiplication by scalars.

(b) Norm of a vector.

(c) We want to be able to compare the norms of the vectors, i.e. we want to refer to the relation $\leqslant$ on the reals.

(d) We want to quantify over all the elements in the space with norm smaller or equal to 1.

We then can define the following signature to reflect those operations:

- $\mathcal{F}$ contains the following symbols of functions:
  - The function $(x + y)$ with arity 2 and true sort.
  - For every real number r, the function symbol $r(x)$ with arity 1 and true sort space.
  - A function symbol $\|x\|$ with arity 1 and sort space the reals with the usual metric.
  - A function symbol $(\|x\|, \|y\|)$ with arity 2 and sort space the real plane with the metric of the max.

- $\mathcal{P}$ contains a predicate symbol $(x \leqslant y)$ of arity 1, and sort space $\mathbb{R}^2$ with the usual metric of the max.

- For every integer n, $\mathcal{AP}$ contain a predicate $(x \leqslant y)_n$ of arity 1 and sort space $\mathbb{R}^2$ with the usual metric of the max (those are the approximations of the relation $x \leqslant y$). The intuitive interpretation of $(x \leqslant y)_n$ will be the relation $(x \leqslant y + 2/n)$.

- $\mathcal{K}$ contains a predicate symbol $B_1(\cdot)$ with arity 1 (and true sort). □

We are now ready to define the terms of the language L_A.

2.2. Terms in L_A

We define terms and sort spaces of the terms by induction.

Definition 1: Terms in L_A

- The variables x_i are terms. Their arity is 1. Their sort is the true sort.

- Given a vector of variables $\vec{x}$ with arity $a < \omega$, and a function symbol f of arity a and sort space (M_f, ρ_f), $f(\vec{x})$ is a term of arity a and sort space (M_f, ρ_f).

- Given a vector of variables $\vec{x}$ with arity $a < \omega$, and a function symbol f of arity a and true sort space, $f(\vec{x})$ is a term of arity a and true sort space.

- Given a collection of terms $\{t_\beta \mid \beta \leqslant \alpha\}$ of arity a_β and true sort space, and a function symbol f of arity α and sort space (M_f, ρ_f), $f(\vec{t})$ is a term with arity the cardinality of the collection of variables in $\{t_\beta \mid \beta < \alpha\}$ and sort space (M_f, ρ_f).

- Given a collection of terms $\{t_\beta \mid \beta \leqslant \alpha\}$ of arity a_β and true sort space, and a function symbol f of arity α and true sort space, $f(\vec{t})$ is a term with arity the cardinality of the collection of free variables in $\{t_\beta \mid \beta < \alpha\}$, and true sort space. □

In summary, terms can have any finite arity and they are of two different types: the ones that have sorts and are going to be interpreted as maps from the model to the sort space, and the ones that do not have sort and are going to be interpreted as maps from the model to the model.

2.3. Formulas in L_A

We are now ready to give the definition of the formulas of L_A by induction as follows.

Definition 2: Atomic formulas. An atomic formula would be any expression of the form:

$$C(\vec{t})$$

where C is a predicate in $\mathcal{P}$ and $\vec{t}$ is a vector of terms such that the arities and sorts agree. □

Definition 3: Formulas in L_A

- An atomic formula is a formula in L_A.
- If $\phi_1, \phi_2, \ldots, \phi_i, \ldots$ $(i < \omega)$ is a collection of formulas in L_A then
$$\bigwedge_{i=1}^{\infty} \phi_i$$
is also a formula in L_A.
- If ϕ is a formula in L_A then $\neg\phi$ is also a formula in L_A.
- Consider a formula $\phi((\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_j, \ldots), \vec{x})$ in L_A. For every $i < \omega$ let a_i be the arity of $\vec{v}_i$, and let K_i be an element of $\mathcal{K}$ of arity a_i. Then the following formula is also in L_A:
$$\exists(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_j, \ldots) \Big(\bigwedge_{j=1}^{\infty} K_j(\vec{v}_j) \wedge \phi((\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_j, \ldots), \vec{x}) \Big).$$

□

Let us remark that the formulas in L_A only involve predicates from the collection $\mathcal{P}$. The collection $\mathcal{AP}$ is not mentioned in this definition, it will be part of the definition of the approximate formulas.

Notice also that we allow quantification over an infinite number of variables. This enhances the expressive power of L_A.

Example 2: Some formulas in L_A

- The map $T{:}(X,\rho) \mapsto (Y,\zeta)$ is continuous in the set K:

$$\forall x \;\; K(x) \Rightarrow \bigwedge_{n=1}^{\omega} \bigvee_{m=1}^{\omega} \Big(\forall y \; K(y) \Rightarrow \big(\rho(x,y) < 1/m \Rightarrow \zeta(T(x),T(y)) < 1/n\big)\Big).$$

- The bounded set K spans an infinitely dimensional subspace in the normed space $(X, \|\cdot\|)$:

$$\forall x(K(x) \Rightarrow \|x\| \leqslant M) \wedge \bigvee_{i=1}^{\omega} \exists \vec{x}_i \,\Big(\bigwedge_{j=1}^{\infty} K(x_j) \wedge \bigwedge_{j=1}^{\infty} \bigwedge_{\substack{n=1\\ n\neq j}}^{\infty} \|x_j - x_n\| \geqslant 1/i\Big).$$

- The set K is compact in the metric space (X,ρ):

$$\forall \vec{x} \,\Big(\bigwedge_{i=1}^{\omega} K(x_i) \Rightarrow \exists y \,\big(K(y) \wedge \bigwedge_{k=1}^{\omega} \bigwedge_{n=1}^{\omega} \bigvee_{m>n}^{\omega} \rho(x_m, y) < 1/k\big)\Big).$$

□

Our next step is to define the semantics for L_A.

2.4. Semantics for L_A

The definitions of the structures are a natural generalization of Henson's notion of a Normed space structure (see [11]).

Fix a collection T of metric spaces (the sort spaces).

Definition 4: Model

Fix a signature Φ for T. A model $\mathcal{A}$ for Φ is a collection

$$\mathcal{A} = (X, \{d^i \mid i < \omega\}, F, P, K)$$

where:

▶ The set X is a fixed set such that for all $i < \omega$: (X^i, d^i) is a metric space with the property, for every $i > 1$, that the topology in X^i induced by d^i is the product topology generated by the topology on (X, d^1).

▶ F=$\{f^\bullet \mid f \in \mathcal{F}\}$ with the property that: for every $f \in \mathcal{F}$ with arity a_f and sort space (M_f, ρ_f) the interpretation $f^\bullet$ is a continuous function going from (X^{a_f}, d^{a_f}) to (M_f, ρ_f).

For every $f \in \mathcal{F}$ with arity a_f and true sort space, the interpretation $f^\bullet$ is a continuous function from (X^{a_f}, d^{a_f}) to (X, d).

▶ $P = \{C^\bullet \mid C \in \mathcal{P}\}$ with the property that: for every $C \in \mathcal{C}$ with sort space (M_C, ρ_C) (hence with arity 1 by the definition of signatures, see Section 2.1) the interpretation $C^\bullet$ is a closed set in (M_C, ρ_C).

For every $C \in \mathcal{C}$ with arity a_C and true sort space, the interpretation $C^\bullet$ is a closed set in (X^{a_C}, d^{a_C}).

▶ The interpretation of the approximate predicates is the natural one.

– For every integer n and every predicate C in $\mathcal{P}$ with arity a and sort space (M_C, ρ_C),

$$C^\bullet_n = \{x \in M_C \mid \exists y \in C^\bullet \ \rho_C(x, y) \leqslant 1/n\}.$$

– For every integer n and every predicate C in $\mathcal{P}$ with arity a and true sort space,

$$C^\bullet_n = \{\vec{x} \in X^a \mid \exists \vec{y} \in C^\bullet \ d^a C(\vec{x}, \vec{y}) \leqslant 1/n\}.$$

▶ $K = \{K^\bullet \mid K \in \mathcal{K}\}$ with the property that: for every $K \in \mathcal{K}$ with arity a_K the interpretation $K^\bullet$ is a closed set in (X^{a_K}, d^{a_K}). The set $K^\bullet$ has the additional property that it has a finite diameter on the metric space (X^{a_K}, d^{a_K}), that is, there exists an integer M such that:

$$\text{Diameter}(K^\bullet) \equiv \sup_{\vec{x},\vec{y}\in K^\bullet} \{d^{a_K}(\vec{x},\vec{y})\} \leqslant M. \qquad \square$$

Once the interpretation for the symbols of Φ are defined for a fixed model $\mathcal{A}$, then one can define the interpretation of the terms in the model, $t^\bullet$, in the obvious way.

When the model we are dealing with and the interpretations of the elements of the language are clear from the context we will drop the symbol $^\bullet$.

The definition of truth follows in the standard way.

Definition 5: Validity

Fix a signature Φ for T. Fix a model $\mathcal{A}$ of Φ. The truth ($\models$) relation in the model $\mathcal{A}$ is constructed in the obvious way from the truth definition for the Atomic case:

> *Let $C(t(\vec{x}))$ be an atomic formula. Let $\vec{a}$ be a vector of the same arity n that $\vec{x}$, $\vec{a} \in (X^n, d^n)$.*
> *Then $\mathcal{A} \models C(t(\vec{a}))$ iff it is true that $t^\bullet(\vec{a}) \in C^\bullet$.* $\square$

Let us give some examples of models in L_A.

Example 3: Models of L_A

- ***The standard model of a metric space (X, d) for a signature Φ.***

Let T be an arbitrary collection of metric spaces. Let $(\mathcal{F}, \mathcal{P}, \mathcal{AP}, \mathcal{K})$ be a signature for T.

A model $\mathcal{A} = (X, \{d^i \mid i < \omega\}, F, P, K)$ is *standard* for Φ if and only if the collection K is exactly equal to the collection of all compact sets in the family of metric spaces $\{(X^i, d^i) \mid i \leqslant \omega\}$.

The standard model is related to the standard neometric family (see [2]) and to the models studied by Anderson in [1].

▶ ***Normed space structures*** (Henson).

Let T containing only the reals with the usual metric. Let $(\mathcal{F}, \mathcal{P}, \mathcal{AP}, \mathcal{K})$ be a signature for T. Furthermore, suppose that for every function symbol f in $\mathcal{F}$ with arity a, for every bounding predicate K in $\mathcal{K}$ with arity a, for every $\epsilon > 0$ we select a real number $\delta(f, K, \epsilon) > 0$.

A model $\mathcal{A} = (X, \{d^i \mid i < \omega\}, F, P, K)$ of Φ is a *normed space structure* with respect to the collection of reals $\{\delta(f, K, \epsilon) \mid f \in \mathcal{F}, K \in \mathcal{K}, \epsilon \in \mathbb{R}\}$ if and only if:

(a) X is a normed space over the field of reals, and the metric d coincides with the norm. Furthermore, the interpretation of $\mathcal{F}$ contains functions that make X a vector space (i.e. sum of two vectors, and for every real number r the function that multiplies a vector by r). The interpretation of $\mathcal{F}$ also contains the norm of X as a function from X to the reals.

(b) The interpretation of $\mathcal{P}$ contains all the closed subsets of the reals.

(c) Every interpretation of a function symbol f with true sort space and arity a is a uniformly continuous function $f : X^a \mapsto X$ with respect to every bounding predicate K with arity a in the following sense:

$$\forall \vec{b} \in K \;\; \forall \epsilon > 0 \quad f(B_{\delta(f,K,\epsilon)}(\vec{b})) \subseteq B_\epsilon(f(\vec{b}))$$

where $B_r(\vec{c})$ denotes the ball centered in $\vec{c}$ and with radius r.

(d) Every interpretation of a function symbol f, with sort space the real numbers and arity a, is a uniformly continuous function $f : X^a \mapsto \mathbb{R}$ on every bounding predicate K with arity a in the following sense:

$$\forall \vec{b} \in K \;\; \forall \epsilon > 0 \quad f(B_{\delta(f,K,\epsilon)}(\vec{b})) \subseteq B_\epsilon(f(\vec{b})).$$

(e) The interpretation of $\mathcal{K}$ is the collection

$$\Big\{ \{\vec{x} \in X^i \; : \; d^i(\vec{0}, \vec{x}) \leqslant r\} \text{ where } r \in \mathbb{R} \text{ and } i \text{ is a positive integer} \Big\}.$$

The normed space structures are introduced in [11] and extensively studied in [11, 6, 10].

▶ ***Nonstandard hulls of metric spaces*** (Fajardo & Keisler).

Fix a nonstandard universe $(V(\Xi), V({}^*\Xi), {}^*)$. Let us recall the basic facts concerning nonstandard hulls.

Given a *-metric space $\overline{M} = (X, \overline{\rho})$, for every x in $\overline{M}$ let $[x]$ be the equivalence class of all the elements in $\overline{M}$ infinitesimally close to x. Then for every c in $\overline{M}$, we define the nonstandard hull of $\overline{M}$ with respect to c as:

$$H(\overline{M}, c) = \big\{[x] \mid x \in \overline{M} \wedge \overline{\rho}(x, c) \text{ is finite}\big\}$$

and the metric in $H(\overline{M}, c)$ is given by the relation:

$$\rho([x], [y]) = {}^0\overline{\rho}(x, y).$$

Let T be the class of all the nonstandard hulls $H(\overline{M}, c)$. Fix a signature Φ for T.

A model $\mathcal{A} = (X, \{d^i \mid i < \omega\}, F, P, K)$ of Φ is a *nonstandard hull model* if and only if:

(a) $(X, d^1) = H(\overline{N}, c)$ for some *-metric space $\overline{N}$, and some c in $\overline{N}$. Furthermore, for every integer i, $(X^i, d^i) = H(\overline{N^i}, (c, c, \ldots, c))$, where $\overline{N^i}$ denotes the cartesian product on $\overline{N}$ i times, with a *-metric that induces the product topology.

(b) If a function f belongs to F then f is uniformly liftable, i.e. there exists an internal function F such that

$$^0F(Y) = f(y)$$

whenever Y lifts y.

(c) If a predicate C belongs to P (where the sort of C is the space $H(\overline{M}, c)$), then C is the standard part of an internal subset of the galaxy $G(\overline{M}, c)$.

(d) If a predicate C belongs to P and it has arity a and true sort, then C is the standard part of an internal subset of the galaxy $G(\overline{N^a}, c)$.

(e) If a set K belongs to K then K is the standard part of a bounded internal set of the galaxy $G(\overline{N}, a)$.

The models just defined can be seen as particular cases of the huge neometric families introduced and studied in a series of papers by Fajardo & Keisler (see [2, 3, 4]). □

Let us now return to the logic L_A. Our intention is to define a notion of approximate formulas for formulas in L_A.

3. Approximate Formulas in L_A

Our aim here is to define syntactically, for any formula ϕ in L_A, approximate formulas ϕ'. We want these approximations to be first order formulas (i.e. they use only finitely many conjunctions, negations and bounded existential quantification), and to verify that their "limit" is the formula ϕ.

We define then the collection of approximate formulas as follows:

Definition 6: Approximate formulas

We define the collection L_{AP} of *approximate formulas* in L_A as follows:

- For any Atomic formula of the form $C(t)$, $C_n(t) \in L_{AP}$.
- If ϕ_1 and ϕ_2 are formulas in L_{AP} then the formula $\phi_1 \wedge \phi_2$ is also in L_{AP}.
- If ϕ is a formula in L_{AP}, then the formula $\neg\phi$ is in L_{AP}.
- Let ϕ be a formula in L_{AP} with free variables among the collection $\{\vec{v}_i \mid i < \omega\} \vee \{\vec{x}\}$. Then the formula

$$\exists(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_i, \ldots)\Big(\bigwedge_{i=1}^{\infty} K_i(\vec{v}_i) \wedge \phi\big((\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_i, \ldots), \vec{x}\big)\Big)$$

  is in L_{AP}. □

Note that the logic L_{AP} allows countably many conjunctions only at the existential quantifier step of the induction. This implies that L_{AP} is close to $L_{\omega\omega}$. In particular L_{AP} can be seen to be a first order multisorted logic. By the work of Heinrich, Henson, Iovino et al, we know that multisorted positive bounded first order logics have many nice model theoretical properties (see [6, 9, 11]).

Note also that L_{AP} is a generalization of the *positive bounded formulas* introduced by Henson in [9].

The intuition about these approximate formulas is that they should verify that ϕ is equivalent to $\bigwedge_{n=1}^{\omega} \phi^n$. Unfortunately this intuition does not seem to extend to formulas including the negation, since what one would get, assuming that the previous equivalence is true, is that $\neg\phi$ is equivalent to $\bigvee_{n=1}^{\omega} \neg\phi^n$.

A natural solution to this problem would be to modify the definition of approximation such that ϕ is equivalent to $\bigvee_{n=1}^{\omega} \bigwedge_{m=1}^{\omega} \phi_{n,m}$. In this way one could hope to include all the formulas in L_A on the approximation scheme.

This idea can not be carried away in the exact form as stated above, but something very close to it, namely:

$$\phi \equiv \bigvee_{h \in I(\phi)} \bigwedge_{n=1}^{\omega} \phi_{h,n}$$

is possible.

In order to do this, we will define by induction the approximations to a formula ϕ and the set of "paths" $I(\phi)$.

The intuition behind the set of paths $I(\phi)$ is as follows:

For the atomic formula C(t), we can imagine the approximations to it disposed in the following way,

$$C_1(t) \mapsto C_2(t) \mapsto \ldots \mapsto C_n(t) \mapsto \ldots \mapsto C(t).$$

But when we deal with the formula $\neg C(t)$, the picture looks different:

$$\begin{array}{ccccccccc}
\neg C_1(t) & \mapsto & \neg C_1(t) & \mapsto & \neg C_1(t) & \mapsto & \dots & \mapsto & \neg C(t) \\
\neg C_2(t) & \mapsto & \neg C_2(t) & \mapsto & \neg C_2(t) & \mapsto & \dots & \mapsto & \neg C(t) \\
\dots & \mapsto & \dots & \mapsto & \dots & \mapsto & \dots & \mapsto & \neg C(t) \\
\neg C_n(t) & \mapsto & \neg C_n(t) & \mapsto & \neg C_n(t) & \mapsto & \dots & \mapsto & \neg C(t) \\
\dots & \mapsto & \dots & \mapsto & \dots & \mapsto & \dots & \mapsto & \neg C(t).
\end{array}$$

In other words, when the formulas involve negations, there can be many different paths that an approximation may follow. In order to deal with this fact, we need to define two indexes (h, n) for the approximate formulas. That is, for every formula ϕ, we will read $\phi_{h,n}$ to be the n-approximation of ϕ along the "path" h.

This implies then that on the definition of the approximations for a formula ϕ we will need to define simultaneously the collection of paths $I(\phi)$ that an approximation can take.

With this in mind, let us give the fundamental definition of this paper.

Definition 7: Approximation of formulas in L_A

For any formula $\phi(\vec{x})$ in L_A we will define sets $I(\phi(\vec{x}))$, and for any $h \in I(\phi(\vec{x}))$ and $n \in \omega$ a formula $\phi_{h,n}(\vec{x}) \in L_{AP}$ in the following way:

▶ *Atomic.* — For any atomic formula $C(t(\vec{x}))$,

– The set $I(C(t(\vec{x}))) = \{\varnothing\}$.

– For every h in $I(C(t(\vec{x})))$, for every integer n,

$$\Big(C(t(\vec{x}))\Big)_{h,n} \equiv \; C_n(t(\vec{x})).$$

▶ *Conjunction.* — For any countable collection of formulas in L_A:

$$\phi_1(\vec{x}), \phi_2(\vec{x}) \dots, \phi_i(\vec{x}), \dots$$

we have that:

– $I(\bigwedge_{i=1}^{\infty} \phi_i(\vec{x})) = \prod_{i=1}^{\infty} I(\phi_i(\vec{x}))$

– For every h in $I(\bigwedge_{i=1}^{\infty}\phi_i(\vec{x}))$, for every integer n,

$$\Big(\bigwedge_{i=1}^{\infty}\phi_i(\vec{x})\Big)_{h,n} \equiv \bigwedge_{i=1}^{n}(\phi_i)_{h(i),n}.$$

▶ *Negation.* — For any formula $\phi(\vec{x})$ in L_A, we have:

– $I(\neg\phi(\vec{x})) = \{h = (h_1, h_2) : \omega \mapsto I(\phi(\vec{x}))\times\omega \mid h \text{ is a covering of } \phi(\vec{x})\}$, where h being a *covering* of $\phi(\vec{x})$ means that:

$$\forall g \in I(\phi(\vec{x}) \;\; \exists n \in \omega \quad \phi_{g,h_2(n)}(\vec{x}) \equiv \phi_{h_1(n),h_2(n)}.$$

– For every h in $I(\neg\phi(\vec{x}))$, for every integer n,

$$(\neg\phi(\vec{x}))_{h,n} = \bigwedge_{i=1}^{n}\neg(\phi_{h_1(i),h_2(i)}(\vec{x})).$$

▶ *Existential.* — For every formula $\phi((\vec{v}_1, \vec{v}_2, \dots, \vec{v}_i, \dots), \vec{x})$ for every vector of bounding sets $\vec{K} = (K_1, K_2, \dots, K_i, \dots)$, we have the following:

– $I\Big(\exists(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_i, \dots)(\bigwedge_{i=1}^{\infty}K_i(\vec{v}_i) \wedge \phi((\vec{v}_1, \vec{v}_2, \dots, \vec{v}_i, \dots), \vec{x}))\Big)$
$= I(\phi((\vec{v}_1, \vec{v}_2, \dots, \vec{v}_i, \dots), \vec{x}))$.

– For every h in $I\Big(\exists(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_i, \dots)(\bigwedge_{i=1}^{\infty}K_i(\vec{v}_i)\wedge\phi((\vec{v}_1, \vec{v}_2, \dots, \vec{v}_i, \dots), \vec{x}))\Big)$, for every integer n, we have that

$$\Big(\exists(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_i, \dots)(\bigwedge_{i=1}^{\infty}K_i(\vec{v}_i) \wedge \phi((\vec{v}_1, \vec{v}_2, \dots, \vec{v}_i, \dots), \vec{x}))\Big)_{h,n}$$

is exactly

$$\exists(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_i, \dots)\Big(\bigwedge_{i=1}^{\infty}K_i(\vec{v}_i) \wedge \phi_{h,n}((\vec{v}_1, \vec{v}_2, \dots, \vec{v}_i, \dots), \vec{x})\Big).$$

□

It can be proved (see [13]) that our approximation scheme coincides with Henson's and Fajardo & Keisler's for the formulas that are atomic, or countable conjunction of atomic formulas.

It is also easy to see that when the formula ϕ is constructed using only atomic formulas, countable conjunctions and bounded existential, the number of paths in $I(\phi)$ is exactly one.

Now we are ready to see some examples of approximations of formulas in L_A.

Example 4: Example of approximate formulas

- Formula ϕ:

$$\exists\vec{x}\,\Big(\bigwedge_{j=1}^{\infty} K(x_j) \wedge \bigwedge_{j=1}^{\infty} \bigwedge_{\substack{n=1\\ n\neq j}}^{\infty} \|x_j - x_n\| \geqslant 1/i\Big).$$

This formula states that there is in K a sequence whose members are at a distance bigger or equal that $1/i$. Let us assume that the signature Φ contains in this case a function norm $\|\cdot\|$, with arity 1 and sort the real numbers with the usual metric, and a predicate $(y \geqslant 1/i)$ of arity 1 and sort the reals with the usual metric.

The reader can verify that the collection of paths $I(\phi)$ is exactly:

$$I(\phi) = \prod_{j=1}^{\infty} \prod_{\substack{n=1\\ n\neq j}}^{\infty} \{\varnothing\}.$$

In other words, there is only one possible path for this formula. Using the definition, we have then that for this $h \in I(\phi)$, for every integer m,

$$(\phi)_{h,m} = \exists\vec{x}\,\Big(\bigwedge_{j=1}^{\infty} K(x_j) \wedge \bigwedge_{j=1}^{m} \bigwedge_{\substack{n=1\\ n\neq j}}^{m} \|x_j - x_n\| \geqslant \frac{1}{i} - \frac{1}{m}\Big).$$

▶ Formula $\neg\phi$:

$$\neg\exists\vec{x}\ \Big(\bigwedge_{j=1}^{\infty} K(x_j) \wedge \bigwedge_{j=1}^{\infty} \bigwedge_{\substack{n=1\\ n\neq j}}^{\infty} \|x_j - x_n\| \geqslant \frac{1}{i}\Big).$$

This formula states that for every sequence in K there exists two distinct elements whose distance is smaller that $1/i$. This is easily seen to be equivalent to the fact that there are only finitely many elements in K whose distance is bigger that $1/i$.

Let us assume again that the signature Φ contains in this case a function norm, with arity 1 and sort the real numbers with the usual metric, and a predicate $(y \geqslant 1/i)$ of arity 1 and sort the reals with the usual metric.

The reader can verify that the collection of paths $I(\neg\phi)$ is exactly:

$$I(\phi) = \big\{f = (f_1, f_2)\ :\ \omega \mapsto I(\phi) \times \omega \mid f \text{ is a covering of } \phi\big\}$$

where the meaning of f being a covering of ϕ is just that:

$$\forall g \in I(\phi)\ \exists n \in \omega\ \ (\phi)_{g,f_2(n)} \equiv (\phi)_{f_1(n),f_2(n)}.$$

In this case, things are very easy because we know that $I(\phi)$ contains only one path. This easily implies that every function $f = (f_1, f_2) : \omega \mapsto I(\phi) \times \omega$ is a covering of ϕ.

Using the definition, we have then that for every function

$$f = (f_1, f_2) : \omega \mapsto I(\phi) \times \omega$$

and for every integer m,

$$(\neg\phi)_{f,m} = \bigwedge_{s=1}^{m} \neg\exists\vec{x}\ \Big(\bigwedge_{j=1}^{\infty} K(x_j) \wedge \bigwedge_{j=1}^{f_2(s)} \bigwedge_{\substack{n=1\\ n\neq j}}^{f_2(s)} \|x_j - x_n\| \geqslant \frac{1}{i} - \frac{1}{f_2(s)}\Big)$$

this can be rewritten in the following simplified form:

$(\neg\phi)_{f,m}$ is equivalent to say that for every $s \leqslant m$ the following is true:

$$\forall \vec{x} \Big(\bigwedge_{j=1}^{\infty} K(x_j) \Rightarrow \bigvee_{j=1}^{f_2(s)} \bigvee_{\substack{n=1 \\ n \neq j}}^{f_2(s)} \|x_j - x_n\| < \frac{1}{i} - \frac{1}{f_2(s)} \Big).$$

□

Let us point out that although there is a calculable procedure to obtain the approximation of every formula ϕ in L_A, the previous example shows that those calculations can be cumbersome. In the coming sections we are going to see that it is a lot easier to obtain the expressions for $\bigwedge_{n=1}^{\omega} \phi_{h,n}$ than for the individual $\phi_{h,n}$.

In the next subsection we are going to introduce the notion of approximate truth.

3.1. Approximate truth

Intuitivelly speaking a formula ϕ is going to be approximately true in a model $\mathcal{A}$ if and only if there exists a path $h \in I(\phi)$ such that every n-approximation of ϕ along this path holds in $\mathcal{A}$.

Definition 8: Approximate validity

Fix a model $\mathcal{A}$ for a signature Φ. Let $\phi(\vec{x})$ be a formula in L_A for this signature. Let a be the arity of $\vec{x}$, let $\vec{b}$ an element of X^a. We say that

$$\mathcal{A} \models_{AP} \phi(\vec{b})$$

if and only if

$$\exists h \in I(\phi(\vec{x})) \ \ \forall n \in \omega \quad \mathcal{A} \models \phi_{h,n}(\vec{b}).$$

Equivalently,

$$\mathcal{A} \models_{AP} \phi(\vec{b})$$

if and only if it is true that

$$\mathcal{A} \models \bigvee_{h \in I(\phi(\vec{x}))} \bigwedge_{n=1}^{\infty} \phi_{h,n}(\vec{b}).$$

□

Let us remark that it can be proved (see [13]) that this definition of approximate truth coincide with Henson's (see [11] for the definition) for the formulas in L_{PB}.

The following proposition is useful, because it will simplify the verification of the approximate truth of formulas without getting involved in the complex computations of the Definition 7. Let us remark that this proposition shows that the notion of approximate truth behaves as expected with respect to the usual logical connectives.

The proof of this propositions is straighforward and can be found in [13].

Proposition 2: Properties of approximate truth

Fix signature Φ and a model $A$of Φ. Let $\phi(\vec{x})$ be a formula in L_A. Then the following is true:

- $\mathcal{A} \models_{AP} \neg\phi(\vec{a})$ *if and only if* $\mathcal{A} \not\models_{AP} \phi(\vec{a})$.
- *For every integer i, let $\phi_i(\vec{x})$ be a formula in L_A. Then the following is true:* $\mathcal{A} \models_{AP} \bigwedge_{i=1}^{\omega} \phi_i(\vec{a})$ *if and only if for every integer i,* $\mathcal{A} \models_{AP} \phi_i(\vec{a})$.
- $\mathcal{A} \models_{AP} \bigvee_{i=1}^{\omega} \phi_i(\vec{a})$ *if and only if there exists an i such that:*

$$\mathcal{A} \models_{AP} \phi_i(\vec{a}).$$

- $\mathcal{A} \models_{AP} (\phi(\vec{a}) \Rightarrow \psi(\vec{a}))$ *if and only if:*

$$\text{if } \mathcal{A} \models_{AP} \phi(\vec{a}) \text{ then } \mathcal{A} \models_{AP} \psi(\vec{a}).$$

Thanks to the previous propositions the manipulation of the approximate truth for formulas is simplified.

4. Elementary properties of approximate truth

Our intention in this section is to take a closer look at the main properties of the notions of approximate formulas and of approximate truth.

The first property states that every set of paths $I(\phi(\vec{x}))$ contains a "dense" countable subset in the following way.

Lemma 3: Dense subset of $I(\phi(\vec{x}))$

Fix a collection T of metric spaces, and a signature Φ over T. Then for every formula $\phi(\vec{x}) \in L_A$, there exists a countable set $D(\phi(\vec{x}))$ such that:

(a) $D(\phi(\vec{x})) \subseteq I(\phi(\vec{x}))$.

(b) *For every h in $I(\phi(\vec{x}))$, for every integer n there exists a path g in $D(\phi(\vec{x}))$ such that:*

$$\phi_{h,n}(\vec{x}) \equiv \phi_{g,m}(\vec{x}).$$

PROOF. — It will be done by induction on the complexity of the formulas in L_A.

▶ *Atomic Formulas.* — Let $C(t(\vec{x}))$ be an atomic formula with free variables in $\vec{x}$.
Then we know that $I(C(t(\vec{x}))) = \{\varnothing\}$ and for every $n \in \omega$ and every $h \in I(C(t(\vec{x})))$:

$$(C(t(\vec{x})))_{h,n} = C_n(t(\vec{x})).$$

Define then: $D(C(t(\vec{x}))) = \{\varnothing\}$.

It is easy to see that those sets verify the conditions (a), (b).

For the connectives and quantifier steps let us assume as induction hypothesis that for formulas ψ of less complexity than the formula $\phi(\vec{x})$, $D(\psi)$ has been defined and verify the properties (a), (b).

▶ *Conjunction.* — Consider the formula

$$\phi(\vec{x}) = \bigwedge_{i=1}^{\alpha} \phi_i(\vec{x}) \in L_A.$$

Recall that:

$$I(\phi(\vec{x})) = \prod_{i=1}^{\omega} I(\phi_i(\vec{x}))$$

and that for all integers n and for all $h \in I(\phi(\vec{x}))$,

$$(\phi(\vec{x}))_{h,n} = \bigwedge_{i=1}^{n} (\phi_i(\vec{x}))_{h(i),n}.$$

The construction of the set $D(\phi(\vec{x}))$ is as follows: for every $n \in \omega$ consider the set $\prod_{i=1}^{n} D(\phi_i(\vec{x}))$. Let

$$p_n : \prod_{i=1}^{\omega} D(\phi_i(\vec{x})) \mapsto \prod_{i=1}^{n} D(\phi_i(\vec{x}))$$

the usual projection map. Since by induction hypothesis for every $i < \omega$, $D(\phi_i(\vec{x}))$ is at most countable, it is always possible to find for every integer n a countable set $Q(n) \subseteq \prod_{i=1}^{\omega} D(\phi_i(\vec{x})) \subseteq I(\bigwedge_{i=1}^{\omega} \phi_i(\vec{x}))$ such that

$$p_n(Q(n)) = \prod_{i=1}^{n} D(\phi_i(\vec{x})).$$

Let then $D(\phi(\vec{x})) = \bigcup_{n=1}^{\omega} Q(n)$.

Using the induction hypothesis it is easy to see that these sets verify the conditions (a), (b).

▶ *Negation.* — Consider the formula $\Psi(\vec{x}) = \neg\phi(\vec{x})$ in L_A.

Recall that:

$$I(\neg\phi(\vec{x})) = \left\{h \in (I(\phi(\vec{x})) \times \omega)^{\omega} \mid h \text{ is } \textit{a covering of}\, \phi(\vec{x})\right\}.$$

Here, *a covering of* $\phi(\vec{x})$ for a function $h = (h_1, h_2) : \omega \mapsto (D(\phi) \times \omega$ means that for all $g \in I(\phi(\vec{x}))$ there exists an integer n such that:

$$(\phi(\vec{x}))_{h_1(n),h_2(n)} = (\phi(\vec{x}))_{g,h_2(n)}.$$

Recall also that for every $n \in \omega$ and every $h = (h_1, h_2) \in I(\neg\phi(\vec{x}))$,

$$(\neg\phi(\vec{x}))_{h,n} = \bigwedge_{i=1}^{n} \neg(\phi(\vec{x})_{h_1(i),h_2(i)}).$$

The construction of the set $D(\neg\phi(\vec{x}))$ is as follows:

By induction hypothesis we know that for every $n \in \omega$ the set

$$(D(\phi(\vec{x})) \times \omega)^{\{1,\dots,n\}}$$

is at most countable since $D(\phi(\vec{x}))$ is at most countable. Let

$$\mathrm{proj}_n : (I(\phi(\vec{x})) \times \omega)^{\omega} \mapsto (D(\phi(\vec{x})) \times \omega)^{\{1,\dots,n\}}$$

the natural projection map. Then it is possible to find for every integer n a countable set $O(n) \subseteq (D(\phi(\vec{x})) \times \omega)^{\omega} \subseteq I(\neg\phi(\vec{x}))$ such that

$$\mathrm{proj}_n(O(n)) = \mathrm{proj}_n(I(\neg\phi(\vec{x}))).$$

Let then $D(\neg\phi(\vec{x})) = \bigcup_{n=1}^{\omega} O(n)$.

Let us verify conditions (a) and (b).

It is clear that $D(\neg\phi(\vec{x}))$ is a countable subset of $I(\neg\phi(\vec{x}))$.

Fix now an integer n and an arbitrary $h \in I(\neg\phi(\vec{x}))$. Then by definition of $D(\neg\phi(\vec{x}))$ there exists an $f \in O(n) \subseteq D(\neg\phi(\vec{x}))$ such that $\mathrm{proj}_n(h) = f$. This implies in particular that for all $i \leqslant n$, $h(i) = f(i)$. We have then:

$$\bigwedge_{i=1}^{n} \neg(\phi(\vec{x})_{h_1(i),h_2(i)}) \equiv \bigwedge_{i=1}^{n} \neg(\phi(\vec{x})_{f_1(i),f_2(i)})$$

that is: $(\neg\phi(\vec{x}))_{h,n} \equiv (\neg\phi(\vec{x}))_{f,n}$.

▶ *Existential.* — Fix a formula ϕ in L_A with free variables among the collection $\{\vec{v}_i \mid i < \beta\} \cup \{\vec{x}\}$. Consider the formula:

$$\psi(\vec{x}) = \exists(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_i, \ldots)\Big(\bigwedge_{i=1}^{\infty} K_i(\vec{v}_i) \wedge \phi((\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_i, \ldots), \vec{x})\Big)$$

Recall: $I(\psi(\vec{x})) = I(\phi((\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_i, \ldots), \vec{x}))$ and $\forall n \in \omega \forall h \in I(\psi(\vec{x}))$,

$$(\psi(\vec{x}))_{h,n} = \exists(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_i, \ldots)\big(\bigwedge_{i=1}^{\infty} K_i(\vec{v}_i) \wedge \phi_{h,n}((\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_i, \ldots), \vec{x})\big)$$

Define then: $D(\psi(\vec{x})) = D(\phi((\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_i, \ldots), \vec{x}))$

The verification of the properties (a), (b) is trivial.

This completes the proof of the Lemma. □

This lemma allows us to prove that the approximate formulas behave nicely enough to guarantee that truth and approximate truth coincide for quantifier free formulas.

We will abbreviate *for all except a finite number of integers* k by $\forall^*$.

Proposition 4: Basic properties of approximations

Fix a collection T of metric spaces. Fix a signature Φ for T. The following statements are true:

(a) *For any formula $\psi(\vec{x})$ in L_A, for any model $\mathcal{A}$ of Φ for any $\vec{b} \in X^{|\vec{x}|}$, for any integer n and for all $F \in I(\psi(\vec{x}))$*

$$\mathcal{A} \models \psi_{F,n+1}(\vec{b}) \Rightarrow \psi_{F,n}(\vec{b}).$$

(b) *For any quantifier free formula $\phi(\vec{x})$ in L_A, for every model $\mathcal{A} = (X, \{d^i \mid i < \omega\}, F, P, K)$ of Φ, for every vector $\vec{b}$ in $X^{|\vec{x}|}$ and every sequence $\vec{f}$ in $X^{|\vec{x}|}$ converging to $\vec{b}$ in the metric $d^{|\vec{x}|}$ we have that:*

$$\mathcal{A} \models \phi(\vec{b})$$

if and only if

$$\exists h \in I(\phi(\vec{x}))\ \ \forall n \in \omega\ \ \forall^* k \quad \mathcal{A} \models \phi_{h,n}(\vec{f}(k)).$$

PROOF. — Straightforward using induction in formulas and the Lemma 3 (page 45). Item (b) follows from item (a). Left to the reader. □

Let us point out to some easy consequences of this proposition.

First, item (b) implies that for quantifier free formulas:

$$\mathcal{A} \models \phi(\vec{b}) \text{ iff } \mathcal{A} \models_{AP} \phi(\vec{b}).$$

This shows that our definition of approximation is *sound*: truth and approximate truth coincide for quantifier free formulas.

Second, item (b) also implies that approximate truth is well behaved with respect to the notion of convergence for quantifier free formulas.

The next example shows that item (b) of the previous proposition is not true for arbitrary formulas in L.

Example 5: Proposition 4 fails in general for existential step.

The essential parts of this example are taken from [5].

The collection T of metric spaces contains only the real numbers with the usual metric. The signature Φ on T consist of

- a symbol of function T with arity 1 and true sort,
- a symbol of function $\|\cdot\|$ of arity 1 and sort the real numbers,
- a symbol of function $(\cdot - \cdot)$ of arity 2 and true sort,
- a predicate C of arity 1 and sort the real numbers,
- a bounding predicate K of arity 1.

Consider the following model $\mathcal{A} = (X, \{d^i \mid i < \omega\}, F, P, K)$ of Φ: $X = \mathcal{C}[0,1]$ the space of continuous real valued functions on $[0,1]$, and the metric d coincide with the supremum norm. For every $i \geqslant 2$ let d^i be any metric that coincide with the product topology in X^i.

Let $T : X \mapsto X$ such that for every $x \in X = \mathcal{C}[0,1]$, $(Tx)(t) = tx(t)$. It is easy to see that this map verifies that for every x, y in X, $\|T(x) - T(y)\| \leqslant \|x - y\|$. Hence this map is continuous.

Let $\|\cdot\|$ be interpreted as the sup norm (easily seen to be continuous).

Let $(\cdot - \cdot)$ be interpreted as the difference between 2 functions in X (trivially continuous).

Let $K = \{x \in X \ : \ 0 = x(0) \leqslant x(t) \leqslant x(1) = 1\}$. Clearly this set is closed and bounded.

Let $C = \{0\}$.

Note first that the map T does not have a fix point in K, i.e. for every x in K, $T(x) \neq x$. If this were not true then it would exist an x in K such that for every $0 \leqslant t \leqslant 1$, $x(t) = tx(t)$: a contradiction.

In other words, we get that:

$$\mathcal{A} \models \neg \exists x (K(x) \wedge C(\|x - T(x)\|)).$$

Finally, for every $\epsilon \geqslant 0$ select any map $x \in K$ with the property that: $\forall t \in [0,1] \ \ \|x(t)\| \leqslant \epsilon$. Then it is easy to see that $\|T(x) - x\| \leqslant \epsilon$.

In other words, we get that:

$$\mathcal{A} \ \models_{AP} \ \exists x (K(x) \wedge C(\|T(x) - x\|)).$$

This completes the example. □

This prompts us to study the models where item 2 of the previous Lemma holds for arbitrary formulas in L_A.

Definition 9: Rich Models

Fix a signature Φ for a collection of metric spaces T, and a model $\mathcal{A} = (X, \{d^i \mid i < \omega\}, F, P, K)$ of Φ.

Let Δ be a collection of formulas in L_A. We say that the model $\mathcal{A}$ is Δ-*rich* if and only if for every formula $\phi(\vec{x})$ in Δ, for every vector $\vec{b} \in X^{|\vec{x}|}$ it is true that:

$$\mathcal{A} \models \phi(\vec{b}) \text{ iff } \mathcal{A} \models_{AP} \phi(\vec{b}).$$

If $\mathcal{A}$ is Δ-rich where Δ is just all the formulas in L_A, we say that $\mathcal{A}$ is a *rich model.* □

We can think of the Δ-rich spaces as spaces where Henson's principle works for Δ.

In order to decide when a model is rich, let us prove the following test.

Theorem 5: Test for rich models

Fix a signature Φ *and a model* $\mathcal{A} = (X, \{d^i \mid i < \omega\}, F, P, K)$*. The following are equivalent:*

- *For every formula* $\psi((\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_i, \ldots), \vec{b})$ with parameters in $\mathcal{A}$, *for every vector of bounding sets* $\vec{K} = (K_1, \ldots, K_i, \ldots)$ *and for every vector of sequences* $\vec{f}$ *in* $\vec{K}$*, if*

$$\exists h \ \forall n \ \forall^* k \quad \mathcal{A} \models \psi_{h,n}(\vec{f}(k), b)$$

then there exists a $(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_i, \ldots)$ *in* K *such that:*

$$\mathcal{A} \models_{AP} \psi((\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_i, \ldots), \vec{b}).$$

- *The model* $\mathcal{A}$ *is rich.*

Proof. — The direction ($\Leftarrow$) is trivial.

($\Rightarrow$) We want to prove that $\mathcal{A} \models \psi$ and $\mathcal{A} \models_{AP} \psi$ are equivalent for every formula ψ. We are going to achieve this by induction on the complexity of the formulas in L_A. By the item (b) of Proposition 4 we already took care of the atomic, conjunction and negation steps. There remains only to prove the existential step.

Assume then that for every formula ϕ with complexity less than the complexity of $\psi((\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_i, \ldots), \vec{y})$ it is true that:

$$\mathcal{A} \models \phi \text{ is equivalent to } \mathcal{A} \models_{AP} \phi.$$

Existential. Let

$$\psi(\vec{x}) = \exists(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_i, \ldots)\Big(\bigwedge_{i=1}^{\infty} K_i(\vec{v}_i) \wedge \phi((\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_i, \ldots), \vec{x})\Big).$$

($\Rightarrow$) Direct.

($\Leftarrow$) Suppose that there exists an $h \in I(\phi((\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_i, \ldots), \vec{x}))$ such that for every integer n we have:

$$\mathcal{A} \models \exists(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_i, \ldots)\Big(\bigwedge_{i=1}^{\infty} K_i(\vec{v}_i) \wedge \phi_{h,n}((\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_i, \ldots), \vec{b})\Big).$$

This implies that there exists sequences $(a(\vec{i}))_n$ in K_i such that for every integer n we have that

$$\forall^* s \in \omega \quad \mathcal{A} \models \phi_{h,n}\big((a(\vec{1})_s, a(\vec{2})_s, \ldots, (a(\vec{i}))_s \ldots), \vec{b}\big)$$

using the hypothesis of the lemma we can affirm then that there exists a vector $(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_i, \ldots) \in \vec{K}$ such that

$$\mathcal{A} \models_{AP} \phi((\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_i, \ldots), \vec{b}).$$

It follows then by induction hypothesis:

$$\mathcal{A} \models \exists(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_i, \ldots)\Big(\bigwedge_{i=1}^{\beta} K_i(\vec{v}_i) \wedge \phi((\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_i, \ldots), \vec{b})\Big).$$

This completes the proof. $\square$

The previous theorem gives us a way to check that a model is rich by focusing on the behavior of sequences of bounding sets elements.

Using the test for rich models, we leave to the reader to verify the following:

Theorem 6: Standard Models are rich

> *Fix Φ a signature, and $\mathcal{A}$ a standard model of a metric space (X, d) for Φ. Then $\mathcal{A}$ is rich.*

A first version of this result, for the notion of approximate truth in formulas that admit finite conjunction, disjunction and bounded quantification, was proved by Anderson ([1]). Fajardo & Keisler obtained a similar approximation result in a logic that admits countable conjunctions and disjunctions using the concept of neoforcing (see [2]).

Let us now check that the nonstandard hulls models are also rich models.

Theorem 7: Nonstandard hulls models are rich

> *Fix Φ a signature and $\mathcal{A}$ a nonstandard model of Φ.*
> *Then $\mathcal{A}$ is L_A-rich.*

Proof. — Recall the definition of nonstandard hull model given in Example 3 (page 34).

From the definition of the nonstandard model, it is easy to prove that for every formula $\phi(\vec{x}, \vec{y})$ in L_{AP}, for every $\vec{b} \in X^{|\vec{y}|}$, for every $K_1, \ldots, K_n \in \mathcal{K}$

$$\{\vec{c} \in \vec{K} : \mathcal{A} \models \phi(\vec{c}, \vec{b})\}$$

is the standard part of an internal set on the galaxy $G(\overline{N^{|\vec{x}|}}, (c, c, \ldots, c))$.

Recall also that if A_n is a chain of internal subsets of $\overline{N}$, then

$$^0(\bigcap_n A_n) = \bigcap_n (^0(A_n)).$$

Now suppose that for a formula $\psi(\vec{x}, \vec{y})$ in L_A, for a vector of bounding sets $\vec{K} = (K_1, K_2, \ldots K_i \ldots)$ there exists an h in $I(\psi)$ such that for every integer n:

$$\exists \vec{a}_n \in \vec{K} \;\; \mathcal{A} \models \psi_{h,n}(\vec{a}_n, \vec{b}).$$

Using Proposition 4, the previous remarks and the fact that countable intersection of internal sets is not empty, it is easy to find an $\vec{a} \in \vec{K}$ such that for every integer n,

$$\mathcal{A} \models \psi_{h,n}(\vec{a}, \vec{b}).$$

This completes the proof of the Theorem. □

5. Concluding remarks

We want to point out that similar studies of the nice properties of nonstandard analysis with respect to some approximation principles have been done already by Fajardo & Keisler (see [2, 3]). The central notion of their study is the neoforcing of a formula instead of the notion of approximate truth of a formula.

In particular, for the notion of neoforcing, and for a collection of structures that generalize the nonstandard hull models (the huge neometric family), they proved an approximation theorem along the same lines that Theorem 7.

The main interest of our approach is that it enables us to develop a model theory of this notion of approximate truth following the footsteps of Henson and his school. For example it can be proved (see [13]) that the logic L_A verifies a compactness principle of the form:

> *For a given family Γ (verifying some mild hypothesis) of models and any sentence ϕ, if there exists a path $h \in I(\phi)$ such that for every integer n there exists a model $\mathcal{A}_n$ in Γ with the property:*
>
> $$\mathcal{A}_n \models \phi_{h,n}$$
>
> *then there exists a model $\mathcal{A}$ in Γ such that $\mathcal{A} \models \phi$.*

On the other hand, in [13] it is proved that the neoforcing can be interpreted in terms of our notion of approximate truth. It is also proved that Theorem 7 can be extended to the huge neometric families, and in general that, under very mild conditions, any structure where truth and neoforcing coincide is a rich structure in our sense. Hence both approaches are equivalent.

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