A DECISION METHOD FOR THE EXISTENTIAL THEOREMS OF NF_2

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§ 1. Introduction.

What is or what is not an existential sentence depends on the language used for a theory. It is natural to formulate NF_2 in the language L with the non-logical signs $\in$, Λ (empty set), $-$ (complement), $\cup$ (union), $\{\ \}$ (singleton set). By the method developped in § 3 of this paper, it is decidable whether any given existential sentence of L is a theorem of NF_2 (THEOREM 3.7). The method is based on the existence of a model M of NF_2 with the particular property of being embeddable in any model of NF_2. It is a direct consequence of this property that an existential sentence of L is a theorem of NF_2 iff it holds in M. Our method actually decides whether a given existential sentence holds in M.

The method developped in § 3 is, however, utterly unusable in pratice. Therefore in § 4, a second method is developped which is a decision method not by itself, but in view of the results of § 3, and which can be readily applied to a great many cases (THEOREM 4.12). For instance, let σ be $\exists x \exists y$ $(x \notin x \ \&\ y \notin y \ \&\ x \in y \ \&\ y \in x)$. Then σ is a theorem of N_2F (just let $x = USC(V)$ and $y = \{USC(V)\}$) but not of NF_2 . Or consider the sentences $\exists x \exists y\ (x \neq y \ \&\ x = \{-\{y \cup \{x\}\}\})$ and $\exists x \exists y\ (x = -y \ \&\ x = \{y \cup \{x\}\})$. Our method shows that the first is a theorem of NF_2 while the second is not (cf. § 5).

§ 2. Some prerequisites.

If formulated in L, NF_2 can be given by the following five axioms (Boffa [1]) : (A1) $\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y$. (A2) $\forall x(x \notin \Lambda)$. (A3) $\forall x(x \in -y \leftrightarrow x \notin y)$. (A4) $\forall x(x \in y \cup z \leftrightarrow x \in y \vee x \in z)$. (A5) $\forall x(x \in \{y\} \leftrightarrow x = y)$. We denote by $\mathbb{N}_0$ the set of the nonnegative integers. We shall construct a model M of NF_2 with the universe $|M| = \mathbb{N}_0$. For all $a,b \in \mathbb{N}_0$, we define $R(a,b)$ (intended to be $\in^M$) to hold iff either of the following conditions holds : (i) b is even and a is an exponent in the binary representation of $\frac{b}{2}$. (ii) b is odd and a is n o t an exponent in the binary representation of $\frac{b-1}{2}$. Then the following can be easily proved (details are given in Oswald [3]) :

2.1. Lemma.

a) $< \mathbb{N}_0, R >$ is a model of NF_2 formulated in $=, \in$.

b) $< \mathbb{N}_0, R >$ can be uniquely expanded to a model of NF_2 .

c) Let M be the unique expansion described in (b). Then M can be isomorphically embedded in every model of NF_2 .

We set $M = < |M|, \in^M, \Lambda^M, -^M, \cup^M, \{\ \}^M >$, where $\in^M$ is R. The following lemma, a direct consequence of 2.1c, is the base for our decision method.

2.2. Lemma.

If σ is an existential sentence of L, then σ is a theorem of NF_2 iff it holds in M.

2.3. Definition.

a) We set $|a| = \{b \in |M| / b \in^M a\}$.

b) We call an individual a of M f i n i t e if $|a|$ is finite, and c o f i n i t e if $|M| \setminus |a|$ is finite.

c) We say that a and b are d i s j o i n t if $|a|$ and $|b|$ are disjoint.

2.4. Lemma.

We let a and b be individuals of M.

a) a is finite or cofinite.

b) $a \in^M a$ iff a is cofinite iff a is odd.

c) If b is finite, and $a \in^{M} b$, then $a < b$.

d) If b is cofinite, and $a \in^{M} b$ does not hold, then $a < b$.

e) To every $M \subset |M|$ such that M or $|M| \setminus M$ is finite, there is exactly one individual a such that $|a| = M$.

f) $a < \{a\}^{M}$.

g) If a is finite, then $-^{M}a = 1 + a$.

h) $-^{M}a < \{a\}$.

i) If $a < b$, then $\{a\}^{M} < \{b\}^{M}$.

j) If a and b are disjoint and finite, then $a \cup^{M} b = a + b$.

k) If a and b are disjoint finite nonempty individuals, then $a < b$ iff $\max |a| < \max |b|$.

Proof.

(a) through (k) are immediate consequences of the definition of R and of 1.3. q.e.d.

By an *individual*, we shall henceforth mean and individual of M. We shall simply say that a formula is *satisfiable* if it is satisfiable in M.

We now let T be any first order theory. By δ_n we denote the formula $\bigwedge_{i<j\leqslant n} x_i \neq x_j$. If p is any predicate sign of L(T), we denote by $\mu(p)$ the number of places of p.

2.5. Definition.

Let P be a finite set of predicate signs of L(T). We call φ a *basic conjunction* with respect to P if φ is a formula of the form $\delta_n \,\&\, \psi$, where ψ is a conjunction satisfying the following conditions : (i) Each factor (we call $\psi_1, \ldots, \psi_n$ the *factors* of $\psi_1 \,\&\, \ldots \,\&\, \psi_n$) of ψ is either $pv_1 \ldots v_{\mu(p)}$ or $\neg pv_1 \ldots v_{\mu(p)}$ for some predicate sign $p \in P$ and variables $v_1, \ldots, v_{\mu(p)} \in \{x_1, \ldots, x_n\}$, or $x_i = t(x_1 \ldots x_n)$ for some $i \leqslant n$ and some term t containing exactly *one* function sign. (ii) For every predicate sign $p \in P$ and any variables $v_1, \ldots, v_{\mu(p)} \in \{x_1, \ldots, x_n\}$, exactly one of $pv_1 \ldots v_{\mu(p)}$ and $\neg pv_1 \ldots v_{\mu(p)}$ is a factor of ψ.

The following theorem is easily by well known methods of predicate logic with identity.

2.6. Reduction theorem.
Let E_p be the set of the existential theorems of T containing no predicate signs other than those of P, and let B_p be the set of the existential theorems of T whose matrix is a basic conjunction with respect to P. Then E_p is decidable iff B_p is decidable.

We denote by $\mathbb{N}$ the set of the positive integers (remember that $\mathbb{N}_o$ is the set of the nonnegative integers). We denote by $F(\mathbb{N})$ the family of the finite subsets of $\mathbb{N}$. For all $I,J \in F(\mathbb{N})$, we define $I \prec J$ to hold iff $\max((I \setminus J) \cup \{0\}) < \max((J \setminus I) \cup \{0\})$. By this definition, the following is true :

2.7. Lemma.
a) $I \prec J$ iff $(I \setminus J) \prec (J \setminus I)$.
b) If I and J are disjoint and nonempty, then $I \prec J$ iff $\max I < \max J$.
c) "$\prec$" is a total ordering on $F(\mathbb{N})$.

We shall let all subscripts run through $\mathbb{N}$. We stick to the convention that $\varphi(x_1 \ldots x_n)$ (or $t(x_1 \ldots x_n)$) means a formula (or a term) all of whose free variables are among $x_1, \ldots, x_n$. For every $n \in \mathbb{N}_o$, we denote by N_n the set $\{i \in \mathbb{N} / i \leqslant n\}$. Notice that N_o is the empty set ϕ.

§ 3. The decidability proof.
3.1. Definition.
Let f be a function with domain D and range R, where $D \subset N_n$ for some $n \in \mathbb{N}_o$ and $R \subset F(\mathbb{N})$.
a) We call f a d m i s s i b l e if $f(i) \subset N_{i-1}$ for every $i \in D$.
b) We call f w e l l a r r a n g e d if f is admissible and satisfies

the following additional conditions : (i) For all $i,j \in D$, if $f(i) \not\prec f(j)$, then $i < j$. (ii) For all i,j such that $i < j$ and $j \in D$, if $k \notin D$ for every k such that $i \leqslant k < j$, then $i \in f(j)$.

The heart of the decidability proof of this paragraph is the fact that every admissible function can be well arranged. In order to make this statement precise, we set $U = D \cup (\cup_{i \in D} f(i))$, and for any permutation p of the elements of U and any $M \subset U$, we let $P(M) = \{p(m)/m \in M\}$.

3.2. Lemma.

To every admissible function f there is a permutation p of the elements of U such that the following is true : (i) $P^{-1}fp$ is well arranged. (ii) For all $i,j \in D$ such that $f(i) = f(j)$, $i < j$ implies $p(i) < p(j)$.

Proof.

We say that an admissible function f is k-arranged if the following three conditions hold : (i) $k \in D \cup \{0\}$. (ii) $f \upharpoonright N_k$ is well arranged. (iii) For every $i \in D \cap N_k$ and every $j \in D \setminus N_k$, either $f(i) \prec f(j)$ or $f(i) = f(j)$. We now describe an algorithm by which, to any k-arranged function f which is not well arranged, a permutation p is constructed such that $P^{-1}fp$ is $(k+\ell)$-arranged for some $\ell > 0$, p preserving the order of any $i,j \in D$ such that $f(i) = f(j)$. Since every admissible function is 0-arranged by definition, it can be well arranged by repeated use of this algorithm.

Now let f be k-arranged but not well arranged. We first note that $D \setminus N_k \neq \phi$, as otherwise f would be well arranged. Let m be the least $i \in D \setminus N_k$ such that $f(i)$ is minimal (with respect to $\prec$) in $\{f(j)/j \in D \setminus N_k\}$, and let $M = (\{m\} \cup f(m)) \setminus N_k$. Let $m_1 < \ldots < m_\ell$ and $n_1 < \ldots < n_r$ be enumerations of M and of $(U \setminus N_k) \setminus M$, respectively (notice that $\ell > 0$). Let further $p_1 < \ldots < p_s$ be an enumeration of U, and $p_1 < \ldots < p_t$ one of $U \setminus N_k$. We let p be the permutation mapping $p_1, \ldots, p_s$ on $p_1, \ldots, p_t, m_1, \ldots, m_\ell, n_1, \ldots, n_r$, in this order. Then, as can be easily proved, $P^{-1}fp$ is $(k+\ell)$-arranged and for all $i,j \in D$ such that $f(i) = f(j)$, if $i < j$, then $p(i) < p(j)$. q.e.d.

We now define the term t_J inductively for every $J \in F(\mathbb{N})$, assuming u_1, u_2, ... to be variables of L.

3.3. Definition.

(i) If $J = \phi$, then t_J is Λ. (ii) If $J = \{j\}$, then t_J is u_j. (iii) $J = \{i,j\}$, where $i < j$, then t_J is $u_i \cup u_j$. (iv) If J has more than 2 elements and $j = \max J$, $J^{-1} = J \setminus \{j\}$, then t_J is $(t_{J^-}) \cup u_j$.

By this definition, if $J = \phi$, $\{5\}$, $\{5,2\}$, $\{3,5,1\}$, $\{2,4,3,1\}$, then we obtain for t_J the terms Λ, u_5, $u_2 \cup u_5$, $(u_1 \cup u_3) \cup u_5$, $((u_1 \cup u_2) \cup u_3) \cup u_4$.

3.4. Lemma.

Let J and K be subsets of N_n, and let $a_1, \ldots, a_n$ be finite individuals.

a) $t_J[a_1 \ldots a_n] < (-t_J)[a_1 \ldots a_n]$.

b) $t_{J \cup K}[a_1 \ldots a_n] = t_J[a_1 \ldots a_n] \cup^M t_K[a_1 \ldots a_n]$.

c) If J and K are disjoint, and $a_1, \ldots, a_n$ are disjoint individuals, then $t_J[a_1 \ldots a_n]$ and $t_K[a_1 \ldots a_n]$ are disjoint.

d) If $a_1, \ldots, a_n$ are disjoint, then $t_J[a_1 \ldots a_n] = \Sigma_{j \in J}\, a_j$.

e) If $a_1, \ldots, a_n$ are disjoint nonempty individuals such that $a_1 < \ldots < a_n$, then $J \prec K$ implies $(-t_J)[a_1 \ldots a_n] < t_K[a_1 \ldots a_n]$.

Proof.

Since $a_1, \ldots, a_n$ are finite, $t_J[a_1 \ldots a_n]$ is finite; hence (a) follows from 2.4g. (b) and (c) are immediate consequences of the definition of t_J and t_K. (d) follows from 2.4j. In order to prove (e), by 2.4b and 2.4g it will suffice to prove that $J \prec K$ implies $t_J[a_1 \ldots a_n] < t_K[a_1 \ldots a_n]$. We omit the parameters $a_1, \ldots, a_n$ and distinguish between three cases: I, II and III. I. $J = \phi$. $t_J = \Lambda^M = 0$. Since $\phi \prec K$, $K \neq \phi$, and since $a_1, \ldots, a_n$ are nonempty, t_K is nonempty, hence $t_K > 0 = t_J$.
II. $J \neq \phi$ and J, K disjoint. By (c) just proved, t_J and t_K are disjoint. As they are nonempty, $t_J < t_K$ iff $\max |t_J| < \max |t_K|$ by 2.4k. Let $j = \max J$ and $k = \max K$. Since $a_1 < \ldots < a_n$ by 2.4k, $\max |t_J| = \max |a_j|$ and $\max |t_K| = \max |a_k|$. Hence $t_J < t_K$ iff $a_j < a_k$ (again by 2.4k) iff $j < k$ iff $J \prec K$ (by 2.7b). III. Now let J and K be arbitrary subsets of N_n

such that $J \prec K$. Then by 2.7a, $(J \setminus K) \prec (K \setminus J)$, and therefore $t_{J \setminus K} < t_{K \setminus J}$ since $J \setminus K$ and $K \setminus J$ are disjoint. By (a) and (b) of this lemma and by 2.4j, $t_J = t_{J \setminus K} + t_{J \cap K}$ and $t_K = t_{K \setminus J} + t_{J \cap K}$. Hence $t_J < t_K$. q.e.d.

We let φ be $\Lambda_{i \leqslant n} \varphi_i$ where each φ_i is either $u_i \neq \Lambda$, or $u_i = \{t_J\}$ or $u_i = \{-t_J\}$ for some $J \subset N_n$. We associate with φ a function f, defined on a subset of N_n, by : $f(i) = J$ if φ_i is $u_i = \{t_J\}$ or $u_i = \{-t_J\}$. f is not defined if φ_i is $u_i \neq \Lambda$. Let f be admissible. Then we define an algorithm, denoted by A, which produces a sequence $a_1, \ldots, a_n$, denoted by $A(\varphi)$, of singletons satisfying φ. The definition of A is by induction on n. If $n = 1$, we let $a_1 = \{\Lambda^M\}^M$. (Notice that φ_1 is $u_1 \neq \Lambda$ since f is admissible.) If $n > 1$, we take for a_n either $\{a_{n-1}\}^M$ or $\{t_J[\, a_1 \ldots a_{n-1}\,]\}^M$ or $\{-^M t_J[\, a_1 \ldots a_{n-1}\,]\}^M$, according as φ_n is $u_n \neq \Lambda$, $u_n = \{t_J\}$, or $u_n = \{-t_J\}$. (Since f is admissible, if φ_n is $u_n = \{t_J\}$ or $u_n = \{-t_J\}$, then $J \subset N_{n-1}$.) We let $g(n)$ be the recursive function defined on $\mathbb{N}$ by : $g(1) = 2$, $g(n) = 2^{n.g(n-1)}$ if $n > 1$.

3.5. Lemma.
Let f be admissible, and let $A(\varphi) = < a_1, \ldots, a_n >$.

a) $\max \{a_1, \ldots, a_n\} \leqslant g(n)$.

b) Assume furthermore that f is well arranged and satisfies the following condition : For all $i,k \in D$ such that $i < k \leqslant n$ and $f(i) = f(k)$, φ_i and φ_k are $u_i = \{t_J\}$ and $u_i = \{-t_J\}$, where $J = f(i)$. Then $a_1 < \ldots < a_n$.

Proof.
In both cases, the proof is by induction on n.

a) $n = 1$: $a_1 = \{\Lambda^M\}^M = 2 \cdot 2^0 = 2 = g(1)$. $n > 1$: $a_n = \{b\}^M$ for some b such that $b \leqslant 1 + \Sigma_{i<n} a_i$. By induction hypothesis (and since g(n) is increasing), $b \leqslant 1 + (n-1)g(n-1)$. By 2.4i, $a_n \leqslant 2 \cdot 2^{1+(n-1)g(n-1)} = 2^{2+(n-1)g(n-1)} \leqslant 2^{n.g(n-1)} = g(n)$.

b) $n = 1$: There is nothing to prove. $n > 1$: By induction hypothesis, $a_1 < \ldots < a_{n-1}$. We distinguish cases I and II, again subdividing II into II_1 and II_2.
I. If φ_n is $u_n \neq \Lambda$, then $a_n = \{a_{n-1}\}^M$, and $a_{n-1} < a_n$ follows from 3.4e.
II. Let φ_n be $u_n = \{t_K\}$ or $u_n = \{-t_K\}$ for some $K \subset N_{n-1}$. II_1. If φ_{n-1}

is $u_{n-1} \neq \Lambda$, then $n-1 \in K$ since f is well arranged. Hence by 3.4d, 3.4a, 2.4h and 2.4i, $a_{n-1} \leqslant \Sigma_{j \in J}\ a_j = t_J[a_1 \ldots a_{n-1}] < -^M t_J [a_1 \ldots a_{n-1}] < \{t_J[a_1 \ldots a_{n-1}]\}^M < \{-^M t_J [a_1 \ldots a_{n-1}]\}^M$, and $a_{n-1} < a_n$ follows. II_2. If φ_{n-1} is $u_{n-1} = \{t_J\}$ or $u_{n-1} = \{-t_J\}$ for some $J \subset N_{n-1}$, then $K \prec J$ would imply $n < n-1$ since f is well arranged. If $J = K$, then by the second condition imposed on f, φ_{n-1} and φ_n are $u_{n-1} = \{t_K\}$ and $u_n = \{-t_K\}$. Hence $a_{n-1} < a_n$ follows from 3.4a and 2.4i. q.e.d.

3.6. Theorem.

Let φ be $\Lambda_{i \leqslant n}\ \varphi_i$, each φ_i being one of $u_i \neq \Lambda$, $u_i = \{t_J\}$, $u_i = \{-t_J\}$ for some $J \subset N_n$. Let f be the function associated with φ as defined above. Then the following are equivalent :

(i) φ is satisfiable by disjoint finite individuals.

(ii) φ is satisfiable by distinct singletons.

(iii) After suitably renumbering $u_1, \ldots, u_n$, f is admissible and for all $i,k \in D$ such that $i < k \leqslant n$ and $f(i) = f(k)$, φ_i and φ_k are $u_i = \{t_J\}$ and $u_k = \{-t_J\}$, where $J = f(i)$.

Proof.

(i) ⇒ (iii) : Let $a_1, \ldots, a_n$ be disjoint finite individuals satisfying φ. Then $a_1, \ldots, a_n$ are distinct since φ requires them to be nonempty. We may assume $a_1 < \ldots < a_n$. Now suppose $j \in J = f(i)$. Then $a_i = \{t_J[a_1 \ldots a_n]\}^M$ or $a_i = \{-^M t_J [a_1 \ldots a_n]\}^M$, hence by 3.4d, $a_j \leqslant \Sigma_{k \in J}\ a_k = t_J[a_1 \ldots a_n]$ and by 3.4a and 2.4f, $a_j < a_i$. Hence $j < i$. Assume further that $i,j \in D$, $i < k \leqslant n$, and $f(i) = f(k) = J$. If φ_i and φ_k were $u_i = \{t_J\}$ and $u_k = \{t_J\}$, or $u_i = \{-t_J\}$ and $u_k = \{-t_J\}$, then $a_i = a_k$ would follow. If φ_i and φ_k were $u_i = \{-t_J\}$ and $u_k = \{t_J\}$, then $a_k < a_i$ would follow by 3.4a and 2.4i, contradicting the assumption that $a_1 < \ldots < a_n$. Hence φ_i and φ_k are $u_i = \{t_J\}$ and $u_i = \{-t_J\}$. (iii) ⇒ (ii) : In view of 3.2, we may assume f to be well arranged. We let $A(\varphi) = \langle a_1, \ldots, a_n \rangle$. Then by the definition of A and by 3.5b, $a_1, \ldots, a_n$ are distinct singletons satisfying φ. (ii) ⇒ (i) : This is true because any distinct singletons are disjoint finite nonempty individuals. q.e.d.

3.7. Theorem.

It is decidable, for any closed existential formula σ of L, whether σ is a theorem of NF_2.

Proof.

In view of 2.2, we have to decide whether, given any open formula $\varphi(x_1,\ldots,x_n)$ of L, $\exists x_1 \ldots \exists x_n \varphi$ is valid in M. Since $t_1 \in t_2$ is equivalent to $(-\{t_1\}) \cup t_2 = -\Lambda$ in NF_2, we may assume that φ does not contain the sign $\in$. By the REDUCTION THEOREM 2.6, we may further assume that φ is a basic conjunction with respect to the empty set of predicate signs, i.e., of the form $\delta_n \,\&\, \chi$, χ being a conjunction of formulas of the form $x_i = t(x_1, \ldots, x_n)$, where t is a term containing exactly one function sign.

It is now convenient to view $x_1, \ldots, x_n$ not a variables but as sets, i.e. we shall henceforth "talk in Th(M)" . We let $m = 2^n$, and we let $u_1, \ldots, u_m$ be the "bits" produced by $x_1, \ldots, x_n$, i.e., the 2^n sets $y_1 \cap \ldots \cap y_n$ where each y_i is x_i or $-x_i$. The sets $u_1, \ldots, u_m$ are a partition of the universe, and for every $i \leqslant n$, there is a $I \subset N_m$ such that $x_i = t_I(u_1 \ldots u_n)$. Every formula $x_i = x_j$, $x_i = \Lambda$, $x_i = -x_j$, $x_i = x_j \cup x_k$ can be replaced by a conjunction of formulas $u_k = \Lambda$. Hence $x_i \neq x_j$ can be replaced by a disjunction of formulas $u_k \neq \Lambda$. The formula $x_i = \{x_j\}$ means that exactly one bit of x_i equals $\{x_j\}$ and the others are empty; so if $x_i = t_I$, $x_i = \{x_j\}$ can be replaced by a disjunction of formulas $u_k = \{x_j\} \,\&\, \Lambda_{\ell \in I\setminus\{k\}}\ u_\ell = \Lambda$, k ranging over I. By a suitable choice of J, we have $x_j = t_J$. If we transform the result of these replacements into disjunctive form, φ becomes a disjunction of formulas each of which is a conjunction of formulas $u_k \neq \Lambda$, $u_k = \Lambda$, $u_k = \{t_J\}$. (This transformation goes back to an idea of V. Ja. Kreĭnovič, cf. KREINOVIC/OSWALD [2]). We may suppose that φ is one of these conjunctions and that for every $k \leqslant m$, at least one of $u_k \neq \Lambda$, $u_k = \Lambda$ is a factor of φ. Our decision method stops here in each of the following cases :

I. if φ is a propositional contradiction (i.e., for some $k \leqslant m$, both $u_k \neq \Lambda$ and $u_k = \Lambda$ are factors);

II. if for some $k \leqslant m$ and some $J \subset N_m$, both $u_k = \Lambda$ and $u_k = \{t_J\}$ are factors.

In both cases, there is no partition of the universe satisfying φ. Otherwise, by a suitable permutation of subscripts φ becomes $(\Lambda_{i \leqslant r} u_i \neq \Lambda)$ & $(\Lambda_{r < i \leqslant m} u_i = \Lambda)$ & ψ, ψ being a conjunction of formulas of the form $u_i = \{t_J\}$, where $i \leqslant r$ and $J \subset N_m$. Since $u_1, \ldots, u_r$ are by themselves a partition of the universe, we omit $\Lambda_{r<i\leqslant m} u_i = \Lambda$ and in each factor $u_i = \{t_J\}$ of ψ, we replace t_J by $t_{J'}$, where $J' = N_r \cap J$. Thus φ becomes $(\Lambda_{i\leqslant r} u_i \neq \Lambda)$ & ψ, ψ being a conjunction of formulas $u_i = \{t_J\}$, where $i \leqslant r$ and $J \subset N_r$. Again, our decision method stops here in each of the following cases :

III. if for some $i \leqslant r$ and some $J, K \subset N_r$ such that $J \neq K$, both $u_i = \{t_J\}$ and $u_i = \{t_K\}$ are factors of ψ. (Notice that $J \neq K$ implies $t_J \neq t_K$);

IV. if for all $i \leqslant r$, ψ has a factor $u_i = \{t_J\}$. (Notice that the union of the u_i's is infinite).

Otherwise for every $i \leqslant r$, ψ has at most one factor $u_i = \{t_J\}$, and there is at least one $i \leqslant r$ such that ψ has no factor $u_i = \{t_J\}$. Let $I = \{i \in N_r / \psi$ has no factor $u_i = \{t_J\}\}$. In M, every finite partition of the universe has exactly one cofinite part. If $u_i = \{t_J\}$, then u_i, being a singleton, is not cofinite. Assume u_i to be the cofinite part in the partition $u_1, \ldots, u_r$. Hence $i \in I$. Delete the factor $u_i \neq \Lambda$. If $u_k = \{t_J\}$ is any factor of ψ such that $i \in J$, then replace t_J by $-t_{J'}$, where $J' = N_r \setminus J$. If this is done for every $i \in I$, each resulting formula φ, possibly after a change of subscripts, satisfies the hypothesis of 3.6. Since 3.6iii states a decidable property of φ, the decision process is complete. q.e.d.

§ 4. A more usable decision method.

Henceforth we let $\psi(x_1, \ldots, x_n)$ be a conjunction of some formulas ψ_{ij}, each ψ_{ij} being either $x_i \in x_j$ or $x_i \notin x_j$.

4.1. Definition.

a) We say that ψ_{ij} and $\psi_{k\ell}$ are s i m i l a r and write $\psi_{ij} \sim \psi_{k\ell}$ if both are atomic or both are negations.

b) We call ψ o r d e r e d if whenever ψ has factors ψ_{ik} and ψ_{jk} such that $k \leqslant i,j$, then $\psi_{ik} \sim \psi_{jk}$.

c) We call ψ o r d e r a b l e if ψ can be ordered by a permutation of subscripts.

d) We say that ψ^- is a r e s t r i c t i o n of ψ and that ψ is an e x t e n s i o n of ψ^- if ψ^- is a conjunction all of whose factors are factors of ψ.

e) We call ψ n - c o m p l e t e if ψ is of the form $\Lambda_{i,j\leq n}\ \psi_{ij}$.

f) We call ψ c o m p l e t e if ψ is n-complete for some n.

Let ψ be n-complete. We denote by $\overline{\psi}$ the matrix whose elements $\overline{\psi}_{ij}$ are $\in$ or $\notin$ according as ψ_{ij} is $x_i \in x_j$ or $x_i \notin x_j$ $(i,j \leq n)$. We call a column of $\overline{\psi}$ h o m o g e n e o u s if it consists solely of $\in$'s or solely of $\notin$'s. We say that c o l u m n s j and k of $\overline{\psi}$ are e q u a l if $\psi_{ij} \sim \psi_{ik}$ for all $i \leq n$. The following lemma states some immediate consequences of these definitions.

4.2. <u>Lemma</u>.

a) ψ is ordered iff every restriction of ψ is ordered.

b) ψ is orderable iff every restriction of ψ is orderable.

c) If ψ is complete and orderable, then $\overline{\psi}$ has a homogeneous column.

If ψ is n-complete and M is a subset of N_n, we denote by ψ_M the formula $\Lambda_{i,j\in M}\ \psi_{ij}$. From 4.2 and the above definitions we conclude :

4.3. <u>Lemma</u>.

If ψ is n-complete, then the following are equivalent : (i) ψ is orderable. (ii) For every $M \subset N_n$, ψ_M is orderable. (iii) For every $M \subset N_n$, $\overline{\psi_M}$ has a homogeneous column.

4.4. <u>Definition</u>.

a) We call $\{a_1, \ldots, a_n\}$ q u a s i - t r a n s i t i v e if for every $i \leq n$, either $|a_i| \subset \{a_1, \ldots, a_n\}$ or $|a_i| \supset |M| \setminus \{a_1, \ldots, a_n\}$.

b) We call ψ r e g u l a r if ψ is complete and orderable and no two columns are equal.

Now we suppose that ψ is n-complete. We let ψ^- be ψ_M for $M = N_{n-1}$. We further let $I(\psi) = \{i < n/\psi_{in}$ is $x_i \in x_n\}$ and $I'(\psi) = \{i < n/\psi_{in}$ is $x_i \notin x_n\}$. We finally let

$$S(\psi;\, a_1,\ldots,a_{n-1}) \;=\; \{b \in |M|/M \models \psi\,[a_1 \ldots a_{n-1}\, b\,]\}.$$

It is clear from this definition that $S(\psi;a_1,\ldots,a_{n-1}) \neq \phi$ implies that ψ^- is satisfied by $a_1, \ldots, a_{n-1}$.

4.5. Lemma.

Let ψ be ordered and n-complete, and let $a_1, \ldots, a_{n-1}$ be individuals satisfying ψ^-. Then the following hold :

a) $S(\psi;a_1, \ldots, a_{n-1})$ is infinite.

b) Let furthermore ψ be regular and $\{a_1,\ldots,a_{n-1}\}$ be quasi-transitive, and let a_n be the least element of $S(\psi;a_1,\ldots,a_{n-1})$. Then a_n is distinct from $a_1, \ldots, a_{n-1}$, and either $|a_n| = \{a_i/i \in I(\psi)\}$ or $|a_n| = |M| \setminus \{a_i/i \in I'(\psi)\}$; and $\{a_1, \ldots, a_n\}$ is quasi-transitive.

Proof.

We omit the parameters of S, I and I'. From the definition of S, it is clear that $b \in S$ holds iff the following three conditions are satisfied : (i) $M \models \psi_{nn}[b]$. (ii) For every $i < n$, $a_i \in^M b$ iff ψ_{in} is $x_i \in x_n$. (iii) For every $j < n$, $b \in^M a_j$ iff ψ_{nj} is $x_n \in x_j$. Condition (ii) is equivalent to (ii') : $\{a_i/i \in I\} \subset |b| \subset |M| \setminus \{a_i/i \in I'\}$. As ψ is ordered, if $j < n$, then $\psi_{nj} \sim \psi_{jj}$. Hence (iii) is equivalent to (iii') : For every $j < n$, $b \in^M a_j$ iff a_j is cofinite. Now (a) holds since (i) and (ii') are satisfied by an infinite number of individuals b and (iii') excludes only a finite number of them. Next we prove (b), assuming that ψ_{nn} is $x_n \notin x_n$ (the proof is quite analogous if ψ_{nn} is $x_n \in x_n$). We let c be the individual determined by $|c| = \{a_i/i \in I\}$ (cf. 2.4e). By the definition of $\in^M$, c is the least b satisfying (i) and (ii'). Assume $j < n$. If a_j is cofinite, then $a_j \neq c$ since c is finite. If a_j is finite, then ψ_{jj} is $x_j \notin x_j$, hence ψ_{nj} is $x_n \notin x_j$. Since ψ is regular by hypothesis, columns j and n are not equal. So there is some $i < n$ such that ψ_{ij} and ψ_{in} are not similar. Hence for this i, exactly one of $a_i \in^M a_j$ and $a_i \in^M c$ holds; so $a_j \neq c$. Therefore, c is distinct from $a_1, \ldots, a_{n-1}$. Moreover, since $\{a_1, \ldots, a_{n-1}\}$ is quasi-transitive, either $|a_i| \subset \{a_1, \ldots, a_{n-1}\}$ or $|a_i| \supset |M| \setminus \{a_1, \ldots, a_{n-1}\}$

for every $i < n$. It follows that $c \in^{M} a_i$ iff a_i is cofinite. Hence c satisfies also (iii'), so $c = a_n$. Finally, from the definition of c and the hypothesis that $\{a_1, \ldots, a_{n-1}\}$ is quasi-transitive, it immediately follows that $\{a_1, \ldots, a_n\}$ is quasi-transitive. q.e.d.

Let ψ be ordered and n-complete. We define an algorithm, denoted by B, which produces a sequence $a_1, \ldots, a_n$, denoted by $B(\psi)$, such that ψ is satisfied by $a_1, \ldots, a_n$. The definition is by induction on n. If $n = 0$ (meaning that ψ is the empty conjunction), then we let $B(\psi)$ be the empty sequence. If $n > 0$ and $B(\psi^-) = < a_1, \ldots, a_{n-1} >$, we let a_n be the least element of $S(\psi;a_1,\ldots,a_{n-1}) \setminus \{a_1, \ldots, a_{n-1}\}$ (this set being nonempty by 4.5a). We then set $B(\psi) = < a_1, \ldots, a_n >$.

4.6. Lemma.
Let $< a_1,\ldots,a_n > = B(\psi)$.
a) $a_1, \ldots, a_n$ are distinct.
b) $M \models \psi[a_1 \ldots a_n]$.
c) If ψ is regular, then $\{a_1, \ldots, a_n\}$ is quasi-transitive.

Proof.
In view of 4.5, the proof by induction on n is immediate. q.e.d.

4.7. Theorem.
The following are equivalent :

(i) ψ is orderable.

(ii) ψ is satisfiable.

(iii) ψ is satisfiable by a sequence of n distinct individuals.

Proof.
Clearly, we may assume ψ to be complete. By 4.6a, (i) implies (iii). Trivially, (iii) implies (ii). In order to show that (ii) implies (i), we let $a_1, \ldots, a_n$ be any individuals satisfying ψ. We assume $a_1 \leqslant \ldots \leqslant a_n$. Then by 2.4c and 2.4d, ψ is ordered. q.e.d.

Example.

Let ψ be $x_1 \notin x_1$ & $x_2 \notin x_2$ & $x_1 \in x_2$ & $x_2 \in x_1$. Then, as remarked in the introduction, $N_2F \vdash \exists x_1 \exists x_2\, \psi$ since for $x_1 = USC(V)$ and $x_2 = \{USC(V)\}$, ψ is satisfied, and since the existence of USC(V) is provable in N_2F. However, $\exists x_1\ \exists x_2\, \psi$ is not a theorem of NF_2. To see this, we notice that $\overline{\psi}$ is

$$\begin{pmatrix} \notin & \in \\ \in & \notin \end{pmatrix}$$

By 4.3, ψ is not orderable since $\overline{\psi}$ has no homogeneous column; so by 4.7, ψ is not satisfiable in M. By 2.2, $\exists x_1\ \exists x_2\, \psi$ is not a theorem of NF_2.

4.8. Definition.

Let ψ be n-complete, and let $\chi(x_1 \ldots x_n)$ be a conjunction of formulas of the form $x_i = t(x_1 \ldots x_n)$, each term t containing exactly one of the function signs Λ , $-$, $\cup$, $\{\ \}$. We say that ψ is c o m p a t i b l e with χ if the following four conditions holds : (i) If for some $j \leq n$, $x_j = \Lambda$ is a factor of χ, then for every $i \leq n$, ψ_{ij} is $x_i \notin x_j$. (ii) If for some $j,k \leq n$, $x_j = -x_k$ is a factor of χ, then for every $i \leq n$, ψ_{ij} and ψ_{ik} are not similar. (iii) If for some j, k, $\ell \leq n$, $x_j = x_k \cup x_\ell$ is a factor of χ, then for every $i \leq n$, ψ_{ij} is $x_i \in x_j$ iff ψ_{ik} is $x_i \in x_k$ or $\psi_{i\ell}$ is $x_i \in x_\ell$. (iv) If for some $j,k \leq n$, $x_j = \{x_k\}$ is a factor of χ, then $j \neq k$ and for every $i \leq n$, ψ_{ij} is $x_i \in x_j$ iff $i = k$.

4.9. Lemma.

a) If ψ & χ is satisfied by a sequence of distinct individuals $a_1, \ldots, a_n$, then ψ is compatible with χ.
b) Let ψ be n-complete and compatible with χ. Let $\{a_1, \ldots, a_n\}$ be a quasi-transitive set of individuals such that ψ is satisfied by $a_1, \ldots, a_n$. Then χ is satisfied by $a_1, \ldots, a_n$.

Proof.

a) is an immediate consequence of the axioms A2 through A5.
b) Clearly we may assume χ to be either one of $x_1 = \Lambda$, $x_1 = -x_2$, $x_1 = \{x_2\}$, $x_1 = x_i \cup x_j$ $(i, j \leq n)$. We only prove the last case, which is the most

interesting one. We let $M = \{1,i,j\}$, $P = \{1,i\}$, and $Q = \{i,j\}$. First, we show that

(1) ψ_{11} is $x_1 \notin x_1$ iff ψ_{ii} and ψ_{jj} are $x_i \notin x_i$ and $x_j \notin x_j$, respectively.

For assume that ψ_{11} is $x_1 \notin x_1$ and, e.g., ψ_{ii} is $x_i \in x_i$ (whence $i \neq 1$). Since ψ is compatible with χ, ψ_{1i} and ψ_{i1} are $x_1 \notin x_i$ and $x_i \in x_1$, respectively. Hence by 4.2c, ψ_P is not orderable. In view of 4.3, this implies that ψ is not orderable. But since ψ is, by hypothesis, satisfiable, it is orderable by 4.7. Or assume ψ_{11}, ψ_{ii}, ψ_{jj} to be $x_1 \in x_1$, $x_i \notin x_i$, $x_j \notin x_j$, respectively (whence $i \neq 1$, $j \neq 1$). Then compatibility rules out that ψ_{1i} and ψ_{1j} are $x_1 \notin x_i$ and $x_1 \notin x_j$. Assume ψ_{1i} to be $x_1 \in x_i$. Then it follows that ψ_{i1} is $x_i \in x_1$ (orderability of ψ_P), ψ_{ij} is $x_i \in x_j$ (compatibility), $i \neq j$ (since ψ_{ii} is $x_i \notin x_i$), ψ_{j1} is $x_j \in x_1$ (orderability of ψ_M), ψ_{ji} is $x_j \in x_i$ (compatibility). But now, we again have a contradiction since ψ_Q, and hence ψ, is not orderable. Now we have to prove that for every individual a,

(2) $a \in^M a_1$ holds iff at least one of $a \in^M a_i$, $a \in^M a_j$ holds.

Compatibility implies that (2) holds for every $a \in \{a_1, \ldots, a_n\}$. It remains to prove (2) for $a \notin \{a_1, \ldots, a_n\}$. If a_1 is finite, then by (1) and 2.4b, a_i and a_j are both finite. Since $\{a_1, \ldots, a_n\}$ is quasi-transitive, $|a_1|$, $|a_i|, |a_j|$ are all subsets of $\{a_1, \ldots, a_n\}$; hence (2) holds also for $a \notin \{a_1, \ldots, a_n\}$. If a_1 is cofinite, then by (1) and 2.4b, one of a_i and a_j, say a_i, is cofinite. Quasi-transitivity implies that $|a_1|$, $|a_i| \supset |M| \setminus \{a_1, \ldots, a_n\}$. Hence (2) holds since for every $a \notin \{a_1, \ldots, a_n\}$, $a \in^M a_1$ and $a \in^M a_i$. q.e.d.

4.10. Definition.

We call $\{a_1, \ldots, a_n\}$ e x t e n s i o n a l if for all $i,j \leq n$ such that $i \neq j$, there is some $k \leq n$ such that exactle one of $a_k \in^M a_i$ and $a_k \in^M a_j$ holds.

4.11. Lemma.

For every individual a, $\{b \in |M| / b \leq a\}$ is extensional.

Proof.

We let $M_a = \{b \in |M| / b \leqslant a\}$ and assume b_1, b_2 to be distinct members of M_a. We distinguish between two cases and use 2.4c and 2.4d.

I. Let, e.g., b_1 be finite and b_2 be cofinite. If $b_1 < b_2$, then not $b_2 \in^M b_1$, but $b_2 \in^M b_2$. If $b_2 < b_1$, then $b_1 \in^M b_2$ but not $b_1 \in^M b_1$.

II. If b_1 and b_2 are both finite or both cofinite, let c be any individual such that, e.g., $c \in^M b_1$ but not $c \in^M b_2$. Then $c < b_1$ if b_1 is finite, and $c < b_2$ if b_2 is cofinite, hence $c \in M_a$ in any case. q.e.d.

We let h be the recursive function defined by $h(n) = 2^{h'(n)} \cdot g(2^{h'(n)})$, where $h'(n) = (n+1)(n+2)$ and where g is the recursive function defined in § 3. In view of the REDUCTION THEOREM 2.6 and since any basic conjunction with respect to $\{\in\}$ is of the form ψ & χ & δ_n, the following theorem again implies that the set of the existential theorems of NF_2 is decidable.

4.2. Theorem.

The existential sentence $\exists x_1 \ldots \exists x_n(\psi \,\&\, \chi \,\&\, \delta_n)$ is a theorem of NF_2 iff there is a number $m \leqslant h(n)$ and a regular extension $\psi^+(x_1 \ldots x_m)$ of ψ such that ψ^+ is compatible with χ.

Proof.

We denote $\exists x_1 \ldots \exists x_n(\psi \,\&\, \chi \,\&\, \delta_n)$ by σ.

1. Let ψ^+ be any regular extension of ψ which is m-complete and compatible with χ. We let $B(\psi^+) = \langle a_1, \ldots, a_m \rangle$. By 4.6, $a_1, \ldots, a_m$ are distinct and satisfy ψ^+, and $\{a_1, \ldots, a_m\}$ is quasi-transitive. By 4.9b, $a_1, \ldots, a_m$ also satisfy χ. Hence ψ & χ & δ_n is satisfied by $a_1, \ldots, a_n$. Thus σ holds in M; so $NF_2 \vdash \sigma$ by 2.2.

2. Conservely, suppose that σ is a theorem of NF_2. We first prove the existence of ψ^+ without paying attention to the upper bound for m. Since $NF_2 \vdash \sigma$, $M \models \sigma$ by 2.2. Let $a_1, \ldots, a_n$ be any individuals satisfying ψ & χ & δ_n. We let

$$(1) \qquad r = \max \{a_1, \ldots, a_n\},$$

and we let $a_{n+1}, \ldots, a_{r+1}$ be an enumeration of $\{a \in |M| / a \leq r\} \setminus \{a_1, \ldots, a_n\}$. Let $m = r + 1$, and let $\psi^+(x_1 \ldots x_m)$ be the uniquely determined extension of ψ which is satisfied by $a_1, \ldots, a_m$. Since $\{a_1, \ldots, a_m\} = \{b \in |M| / b \leq a_m\}$, the set $\{a_1, \ldots, a_m\}$ is extensional by 4.11. Hence ψ^+ is regular. Since $x_{n+1}, \ldots, x_m$ do not occur in χ, χ is also satisfied by $a_1, \ldots, a_m$. By 4.9a, ψ^+ is compatible with χ.

We finally show that in (1), one may assume that

$$r \leq h(n) - 1. \tag{2}$$

To see this, we inspect the proof of THEOREM 3.7 and apply 3.5a. In the proof of 3.7, each factor $x_i \in x_j$ or $x_i \notin x_j$ of ψ is replaced by $(-\{x_i\}) \cup x_j = -\Lambda$ and $(-\{x_i\}) \cup x_j \neq -\Lambda$, respectively. The result is then transformed into a disjunction of formulas of the form

$$\exists x_{n+1} \cdots \exists x_k (\chi' \,\&\, \delta_k), \tag{3}$$

where χ' is a formula of the same type as χ but may contain any of the variables $x_1, \ldots, x_k$. Since $\psi \,\&\, \chi \,\&\, \delta_n$ is satisfiable, at least one the formulas (3) is satisfiable; conversely, whenever $a_1, \ldots, a_k$ satisfy $\chi' \,\&\, \delta_k$ in this particular formula, then $a_1, \ldots, a_n$ satisfy $\psi \,\&\, \chi \,\&\, \delta_n$. In the transformation process, we have to introduce new variables $x_{n+1}, \ldots, x_{2n}$ for $\{x_1\}, \ldots, \{x_n\}$, and new variables $x_{2n+1}, \ldots, x_{3n}$ for $-\{x_1\}, \ldots, -\{x_n\}$. We may have to add $x_{3n+1} = \Lambda$ and $x_{3n+2} = -x_{3n+1}$. Whenever ψ_{ij} is $x_i \notin x_j$, we need an additional variable for $x_{2n+i} \cup x_j$ (which stands for $(-\{x_i\}) \cup x_j$). Hence we may assume that in (3), $k \leq n^2 + 3n + 2 = (n+1)(n+2)$. Now let $\chi' \,\&\, \delta_k$ in (3) be satisfiable. Then the proof of 3.7 shows that there is some conjunction φ of the form $\Lambda_{i \leq p} \varphi_i$, where $p = 2^k$ and each φ_i is either $u_i \neq \Lambda$ or $u_i = \Lambda$, or $u_i = \{t_J\}$ for some $J \subset N_p$, satisfiable by a partition of the universe. Conversely, from every partition satisfying this particular φ, a sequence $a_1, \ldots, a_k$ satisfying $\chi' \,\&\, \delta_k$ can be recovered. By 3.5a, there is a partition $b_1, \ldots, b_p$ satisfying φ such that each finite part does not exceed $g(p)$. Let $b_1, \ldots, b_q$ be the finite nonempty parts (whence $q < p$), and let b_{q+1} be the unique cofinite part. From 3.5a, we can actually conclude that $b_i \leq g(q)$ for every $i \leq q$. From 2.4g and 2.4j, we conclude that $b_{q+1} = 1 + \Sigma_{i\leqslant q} b_i$ and that $a_j \leqslant 1 + \Sigma_{i\leqslant q} b_i$ for all $j \leqslant k$, where $a_1, \ldots, a_k$ is the solution of χ' & δ_k recovered from $b_1, \ldots, b_p$. Thus if ψ & χ & δ_n is satisfiable at all, it is satisfiable by individuals not exceeding $1 + q.g(q)$. Now (2) follows from $k \leqslant (n+1)(n+2)$, $p = 2^k$, $q < p$, $r \leqslant 1 + q.g(q)$ and from the fact that $g(i) \geqslant 2$ for every $i \in \mathbb{N}$. q.e.d.

§ 5. Some examples.

In this section, we omit the reference to M in the function signs and use the common notation V for $- \Lambda$.

1) The sentence $\exists x_1 \exists x_2(x_1 \neq x_2 \;\&\; x_1 = \{-\{x_2 \cup \{x_1\}\}\})$ is a theorem of NF_2. In order to see this, we first notice that $x_1 = \{-\{x_2 \cup \{x_1\}\}\}$ is equivalent in NF_2 to $\exists x_3 \exists x_4 \exists x_5 \exists x_6 (x_3 = \{x_1\} \;\&\; x_4 = x_2 \cup x_3 \;\&\; x_5 = \{x_4\} \;\&\; x_6 = -x_5 \;\&\; x_1 = \{x_6\})$. We shall look for a 6-complete regular ψ which is (an extension of the empty conjunction and) compatible with $x_3 = \{x_1\} \;\&\; x_4 = x_2 \cup x_3 \;\&\; x_5 = \{x_4\} \;\&\; x_6 = -x_5 \;\&\; x_1 = \{x_6\}$. Disregarding the factor $x_4 = x_2 \cup x_3$, we see that any 6-complete ψ is compatible with the four remaining factors iff $\overline{\psi}$ is of the following form (the elements marked by dots are arbitrary) :

$$\overline{\psi} = \begin{pmatrix} \notin & . & \in & . & \notin & \in \\ \notin & . & \notin & . & \notin & \in \\ \notin & . & \notin & . & \notin & \in \\ \notin & . & \notin & . & \in & \notin \\ \notin & . & \notin & . & \notin & \in \\ \in & . & \notin & . & \notin & \in \end{pmatrix} .$$

The orderability conditions are $a_6 < a_1$, $a_1 < a_3$, $a_4 < a_5$, $a_4 < a_6$ (by 2.4c and 2.4d). They reduce to $a_4 < a_6 < a_1 < a_3$ and $a_4 < a_5$. We try our luck by arranging the variables $x_1, \ldots, x_6$ in the order $x_4, x_5, x_6, x_1, x_2, x_3$. Then $\overline{\psi}$ becomes

$$\begin{array}{c|cccccc} & x_4 & x_5 & x_6 & x_1 & x_2 & x_3 \\ \hline x_4 & . & \in & \notin & \notin & . & \notin \\ x_5 & . & \notin & \in & \notin & . & \notin \\ x_6 & . & \notin & \in & \in & . & \notin \\ x_1 & . & \notin & \in & \notin & . & \in \\ x_2 & . & \notin & \in & \notin & . & \notin \\ x_3 & . & \notin & \in & \notin & . & \notin \end{array} .$$

After this permutation, ψ has become ordered. (Every column is homogeneous down from the diagonal). We try to complete ψ in a way that it remains ordered and becomes compatible with $x_4 = x_2 \cup x_3$. Since up to now $\overline{\psi}$ has no homogeneous column, we try by taking $\in$ for the column of x_4. Then compatibility with $x_4 = x_2 \cup x_3$ requires us to take $\in$ for the column of x_2 with the exception of $\overline{\psi}_{12}$. Since the column of x_4 consists solely of $\in$'s, we have to take $\notin$ for $\overline{\psi}_{12}$ to make ψ regular. Then by 4.12, we see that the given sentence is a theorem of NF_2. If we want to find individuals satisfying the matrix of the given sentence, we apply the algorithm B to ψ. B produces in turn : $a_4 = V$, $a_5 = \{a_4\} = \{V\}$, $a_6 = -\{a_4\} = -\{V\}$, $a_1 = \{a_6\} = \{-\{V\}\}$, $a_2 = -\{a_1\} = -\{\{-\{V\}\}\}$, $a_3 = \{a_1\} = \{\{-\{V\}\}\}$. Now if the matrix of the given existential sentence is denoted by φ, we have $M \models \varphi[a_1 a_2]$ and $NF_2 \vdash x_1 = \{-\{V\}\}$ & $x_2 = -\{\{-\{V\}\}\} \rightarrow \varphi$. Note that since every individual of M corresponds to a term of L, the algorithm B can be applied to produce terms which in NF_2 provably satisfy the given basic conjunction.

2) If ψ is $x_1 \in x_1$ & $x_1 \notin x_3$ & $x_2 \in x_1$ & $x_3 \notin x_1$ & $x_3 \notin x_2$, and χ is $x_1 = x_2 \cup x_3$, then $\exists x_1\ \exists x_2\ \exists x_3(\psi\ \&\ \chi\ \&\ \delta_3)$ is a theorem of NF_2. It is easy to see that there is only one 3-complete extension of ψ which is orderable and compatible with χ; for this ψ,

$$\overline{\psi} = \begin{pmatrix} \in & \in & \notin \\ \in & \in & \notin \\ \notin & \notin & \notin \end{pmatrix} .$$

Then columns of 1 and 2 are equal, so ψ is not regular. If we introduce a new variable x_4, then by arranging the variables in the order x_4, x_3, x_1, x_2 we get a regular extension which is compatible with χ :

	x_4	x_3	x_1	x_2
x_4	$\notin$	$\in$	$\in$	$\notin$
x_3	$\notin$	$\notin$	$\notin$	$\notin$
x_1	$\notin$	$\notin$	$\in$	$\in$
x_2	$\notin$	$\notin$	$\in$	$\in$

.

Hence by 4.12, $\exists x_1 \exists x_2 \exists x_3(\psi \ \& \ \chi \ \& \ \delta_3)$ is a theorem of NF_2. Applying the algorithm B, we obtain $a_4 = \Lambda$, $a_3 = \{a_4\} = \{\Lambda\}$, $a_1 = -\{a_3\} = -\{\{\Lambda\}\}$, $a_2 = -\{a_3, a_4\} = -\{\{\Lambda\}, \Lambda\}$.

3) $\exists x_1 \exists x_2(x_1 = -x_2 \ \& \ x_1 = \{x_2 \cup \{x_1\}\})$ is not a theorem of NF_2. To see this, we first denote the matrix of the given sentence by χ. Then χ is equivalent in NF_2 to $\exists x_3 \exists x_4 \chi'$, χ' being $x_1 = -x_2 \ \& \ x_3 = \{x_1\} \ \& \ x_4 = x_2 \cup x_3 \ \& \ x_1 = \{x_4\}$. Compatibility requires $\overline{\psi}_{11}$ and $\overline{\psi}_{44}$ to be $\notin$ and $\overline{\psi}_{14}$ and $\overline{\psi}_{41}$ to be $\in$. Thus for $M = \{1,4\}$, ψ_M is not orderable.

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