AXIOMATISING SET THEORY WITH A UNIVERSAL SET

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There are various problems in Set theory with a universal set : chief of them is persuading people that there is anything worth studying. I shall say less on this subject than I would like : you who have already read this far are presumably at least willing to listen. However I will allow myself one gibe : ZF is obviously the core of any sensible axiomatic theory of the wellfounded sets. This is not the same as saying that there are no others. Much of the plausibility of ZF as an axiomatisation of Set Theory comes from mistaking arguments for the first for arguments for the second.
Although it is now over 40 years since the first axiomatic set theory with $V \in V$ was published there is stil no agreement on even a core for an axiomatisation of set theory with $V \in V$. In this paper I present some (I hope) persuasive motivations for some axioms. The programme is best begun by looking at the most basic problem of all namely.

The problem of identity in illfounded set theory.

The problem of identity in set theory with a universal set is the same as the problem of identity in the more general case of illfounded set theory. Indeed I shall not make much of the difference since there seems little motivation for illfounded set theory if one is not interested in a universal set.

The axiom of extensionality

$$(\forall x)(\forall y)(x = y \equiv (\forall z)(z \in x \equiv z \in y))$$

summarises all that conventional wisdom has had to say so far about "=" in set theory. It is the closest we come to saying in any formal sense that sets are that-which-is-extensional. A set is just the collection of its members, that and no more. Thus extensionality, in conjunction with the axiom of Foundation, enables us to decide when $x = y$ by seeing if their members are identical. The regress we are here launched on must terminate because the ranks of the things we look at is reduced by the induction step. I am becoming more and more convinced that the appeal of the axiom of Foundation is simply that it provides us with this elegant recursive characterisation of identity and there by spares us the need to do any deep thinking on the subject of identity of sets. Historically this restriction may have been fruitful as it enabled us to concentrate our efforts and attention on those parts of set theory where results could be obtained quickly and applied widely; recently the profuse growth of parts of wellfounded set theory of no interest to non-set theorists has begun to suggest that is has had all the help it needs. Perhaps the time is now ripe to reopen the fundamental questions we have ignored since the turn of the century.

The problem, then, is that the regress of which I spoke in the last paragraph "$x = y$? Are all their members identical ? Are all their members identical ? ... ?", cannot be relied upon to terminate. Here we can profitably introduce some concepts from game theory. Notation and terminology here will be standard except for use of the word "Wins" with upper-case 'W' to mean "has a winning strategy for" and that a <u>strategy</u> is not a thing that says "When here, do this" but only "When here, do one of these", that is, a non-deterministic strategy. This is because, as I have argued elsewhere [1], AC is probably false in any sensible set theory with a universal set, so that if we use strategies in the standard sense (which can often be nothing more than thinly concealed choice functions) we are liable to find that a game fails to have a winning strategy — for reasons which are nothing to do with the game itself. The first game here will be notated $G_{x=y}$ ("The identity game") to commemorate the fact that it is being played to decide whether or not $x = y$

Player II moves first, choosing a subset R_1 of $x \times y$ such that
$R_1''x = y$ and $\breve{R}_1''y = x$

Player I picks an ordered pair from II's previous choice

Player II picks R_{n+1} a subset of $x_n \times y_n$ (where $<x_n,y_n>$ was I's previous choice) s.t. $R_{n+1}''x_n = y_n$ and $\breve{R}_{n+1}''y_n = x_n$.

Player II loses if she is confronted with x_n, y_n one of which is empty and the other not. I loses if she picks $<x_n,y_n>$ both of which are empty (notice that this allows the existence of urelemente). If the game goes on for ever II wins. The idea is that II is trying to prove $x = y$, and I is trying to prove $x \neq y$. In earlier versions of this paper II had to pick bijections rather than relations, the rationale being that a set cannot have two identical members. The present version is probably better all the same, because it does not compel us to decide, before we start playing, whether or not some things x_1, x_2 in x are identical (which we could discover only by playing $G_{x_1=x_2}$). Another way of putting this would be to say that to specify formally the rules governing II's moves in the game where she has to play bijections would use the "=" symbol whose meaning is explained only by the game which is yet to be played.

Player II's choice of R_{n+1} when faced with x_n , y_n is obtained by partitioning x_n and y_n into equivalence classes under identity and then pairing the equivalence classes for x_n with those for y_n in some appropriate way. On the face of it, this suggests that we should always require II's choice R to satisfy a condition

$$(u \; R \; v \wedge u' \; R \; v \wedge u' \; R \; v') \to u \; R \; v'$$

but it is not hard to persuade oneself that the resulting games are equivalent and the proof is omitted.

$G_{x=y}$ is an open game. That is to say, if player I wins at all, she has done so after finitely many moves. So I or II must have a winning strategy.

It is not hard to see that the relation

$$\text{II Wins } G_{x=y}$$

is an equivalence relation. Indeed it looks like a very good candidate for a definiens of "$x = y$". However there are good reasons for looking for something even stronger. Let us define j, an operator on maps, so that

$$(j\text{'}f)\text{'}X = f\text{''}X$$

and let us define, for each n, an equivalence relation $\sim_n$ by

$$x \sim_n y \text{ iff } (\exists\pi)(\pi \text{ a permutation of } V \wedge (j^n\text{'}\pi)\text{'}x = y)$$

$$x \sim_\infty y \text{ iff } x \sim_n y \text{ for all } n$$

and we invoke the notations for equivalence classes $[x]_n$, $[x]_\infty$ in the usual way. The importance of n-congruence derives from the fact that if $x \sim_n y$ then x and y satisfy the same stratified predicates in which they are of type 0. This derives from a theorem, important in the folklore of NF, that

$$\varphi(x,y,z \ldots) \leftrightarrow \varphi((j^n\text{'}\pi)\text{'}x,\ (j^m\text{'}\pi)\text{'}y,\ (j^k\text{'}\pi)\text{'}z \ldots)$$

where n, m, k ... are the types of x, y, z ... in φ.

This fact, which will not be proved here, will be used later on. Thus $x \sim_n y$ says that the top n "layers" of the transitive closure of x look like the top n layers of the transitive closure of y. This being the case, extensionality would lead us to be very sceptical of the desirability of having x, y such that $x \sim_\infty y$ but $x \neq y$. Since there is no obvious way of constructing a winning strategy in $G_{x=y}$ simply from the fact that $x \sim_\infty y$ a tougher definition of identity will be required.

Consider again the game $G_{x=y}$. Let us suppose I has a winning strategy. Let us consider the tree of all plays obtained by I using her winning strategy and II doing anything legal. This tree is wellfounded since all plays (branches) terminate after a finite number of steps (the use of DC here may or may not be significant — see the next game below where a similar problem occurs) and accordingly has a rank. Let us notate this ordinal $\epsilon_{x,y}$. $\epsilon_{x,y}$ looks rather like a truth-value for "$x = y$" but the idea of ordinals as truth-values of anything is profoundly repugnant and suggests that we have got too much structure here, and that some of it is spurious. Fortunately we have the following crucial fact :

If $\in_{x,y}$ and $\in_{y,z}$ are both infinite, so is $\in_{x,z}$.

Proof.
"$\in_{x,y}$ is infinite" simply says that, for each finite integer n, player II has a strategy that enables her to postpone defeat until after n moves have been played. If II has such a strategy for $G_{x=y}$ and one for $G_{y=z}$ she can get one for $G_{x=z}$ by playing in $G_{x=z}$ the composition of the plays prescribed by her strategies in $G_{x=y}$ and $G_{y=z}$, having arbitrarily assigned, given I's choice of $< u,v >$, a w such that $< u,w >$ is a play for I in $G_{x=y}$ and $< w,v >$ a play for I in $G_{y=z}$. What this means is that the relation

" $\in_{x,y}$ is infinite"

is an equivalence relation. And it will be our explication of "=" in ill-founded set-theory.

Axiom of strong Extensionality.
$(\forall x)(\forall y)(x = y \leftrightarrow (\forall n)$(II has a strategy to postpone defeat in $G_{x=y}$ for n moves)).

Of course this axiom deliberately expunges a lot of structure : If we had defined "$x = y$" as "II Wins $G_{x=y}$ " then we would have lots of exciting equivalence relations to play with, since ω cannot be the only ordinal α such that "$\in_{x,y} \geqslant \alpha$" is an equivalence relation, but we would not have established that ∞-equivalence is identity. A Quine atom is an object $x = \{x\}$. Strong extensionality prevents there being more than one Quine atom. Indeed it prevents there being more than one object whose transitive closure does not contain the empty set. It also excludes the possibility of automorphisms of the universe.

$G_{x=y}$ has generalisations which can be useful when defining identity in models that we obtain by deleting objects, e.g. urelemente, from some initial model. First we identity objects whose symmetric difference consists entirely of things to be deleted. Then we delete all but one of each equivalence class, and iterate. The same effect can be achieved by playing a version of $G_{x=y}$ where the domains and ranges of the relations played by II do not include any objects which are to be deleted.

The next game we consider has a much simpler structure. This game, played with an initial given set x, is notated G_x and is played as follows : I moves first by picking a member of x. Thereafter I and II alternate moves, each picking a member of the preceding player's choice, until the game is ended by one of them trying to pick a member of an empty set (the game may be played in universes with urelemente) and thereby losing. If the game goes on for ever it is a draw. Obviously for some x (such as V) there will be plays in G_x which never terminate (I and II could go on for ever picking V each time) but one has the feeling that empty sets ought to be sufficiently dense in the transitive closure of x for G_x to admit a winning strategy for one player or the other. Let us adopt the definitions

$$I = \{x : I \text{ Wins } G_x\} \qquad II = \{x : II \text{ Wins } G_x\}.$$

Obviously $x \in I$ iff $(\exists y \in x)(y \in II)$, and dually $x \in II$ iff $(\forall y \in x)(y \in I)$. We can rewrite this as $I = \cup B''II$ (where $B'x = \{y : x \in y\}$) and $II = p'I$. If, with a view to readability, we invent a new function letter b so that $b'x = \cup B''x$ (the 'b' is an upside-down 'p' to remind us that b corresponds to $(\exists x \in \ldots)$ and p to $(\forall x \in \ldots)$) we can write it as $I = b'II$ and $II = p'I$). This is rather reminiscent of the fact that $x \in WF$ iff $(\forall y \in x)(y \in WF)$. Apart from the elegant characterisation this enables us to give of I and II in a language whose subformula relation is illfounded it invites us to consider what happens when we stick in extra quantifiers, like, for example

$$\begin{array}{llllll} x \in X \text{ iff} & \exists y \in x & \forall w \in y & \exists u \in w & u \in Y & \\ & Y & \forall & \exists & \forall & X \end{array}$$

or, for short $X = b|p|b'Y$ and $Y = p|b|p'x$. In this case I and II are no longer unique solutions since $b'Y$ for X and $p'X$ for Y will satisfy the same identity. This is rather reminiscent of the way e^x splits into sinhx and coshx when we require not $f = Df$ but merely $f = D^2f$. Both in that case and here we find that by increasing the number of iterations more roots will appear. This parallel will not be explored further in this paper.

Once we notice that $\Lambda \in II$ and $V \in I$ the discussion above suggests the following recursive construction :

$$I_0 = \{V\} \qquad I_{\alpha+1} = \cup B''II_\alpha$$

$$II_0 = \{\Lambda\} \qquad II_{\alpha+1} = p'I_\alpha$$

taking sumsets at limit ordinals.
It is not hard to show by induction that I_α and II_α are increasing sequences under inclusion. Let us associate with each object in I or II its rank — the least α such that it belongs to I_α or II_α. We shall need to show that everything in I or II does in fact have a rank. The proof is analogous to that in ZF that every wellfounded set has a rank.

Suppose $x \in II$ is unranked. Then every $y \in x$ is in I but then some $y \in x$ is unranked, otherwise rank of x is just $\sup_{y \in x}$ (rank'y + 1). Similarly suppose $x \in I$ is unranked. Then there is $y \in x$, $y \in II$. But no such y can be of rank α, otherwise $x \in I_{\alpha+1}$.

This proof of illfoundedness enables (in either case) the "losing" player to construct a strategy ("Play unranked sets !") which results in an infinite play and a draw, contradicting the existence of a winning strategy.

This justifies the definition of I and II as the union of their partial sums over all ordinals.

<u>Fact.</u>
II_α and I_β are disjoint for all α, β.

Suppose α and β are minimal counterexamples, then we have $x \in II_\alpha$, $x \in I_\beta$. So there is $y \in x$ such that $y \in II_\delta$ for some $\delta < \beta$. But any such y (since $y \in x \in II_\alpha$) must also be in I_γ for some $\gamma < \alpha$ contradicting minimality of α, β .

This enables us to construct a canonical strategy for the winning player.

The minimal strategy.
"When confronted with x, play anything in $x \cap II$ of minimal rank".

It is well known that the rank of a wellfounded set x can be defined either as the rank of $\in \restriction TC'x$ considered as a wellfounded relation or as the least ordinal α such that $x \in V_{\alpha+1}$. There is a corresponding result here : let the pseudorank of x be the least ordinal α such that $x \in I_{\alpha+1} \cup II_{\alpha+1}$. This pseudorank of x is the same as the rank of the tree of plays obtainable in G_x by the Winning player using the minimal strategy and the other player doing anything legal. The proof is an easy induction on rank and is left to the reader. The reader may also wish to verify that any wellfounded set of rank α must have pseudorank α too. The proofs all have such an engaging familiarity that it suggests one should adopt, as an analogue of the axiom of foundation the following

Axiom of $\in$-determinacy.
$V = I \cup II$.

(There is a slight blemish to the parallel between the axioms of $\in$-determinacy and foundation, namely that $\in$-determinacy tells us that we can associate with each set x a canonical tree which is wellfounded in the weak sense that every path through it is finite. This involves DC in subsequent proofs. We could frame $\in$-determinacy in a way that gets round this by defining recursively, on the tree of possible plays in G_x, a function that takes two values, 0 and 1 where "$f'y = 1$" says "I has a Winning strategy from stage y" and "$f'y = 0$" says "II has a winning strategy from stage y". The new version of $\in$-determinacy would then say that for all x, this function is defined on the whole of the tree of plays of G_x).

To lend plausibility to this axiom, we can prove it for a large class of sets :

<u>Theorem</u>.

Let X be n-symmetric, n even (odd) then

either I (II) Wins G_x in n+2 moves
or II (I) Wins G_x in n+3 moves .

Let us consider only the case n = 6, to keep the formulae readable. Then let "$\varphi(B'\Lambda,X)$" be an abbreviation for

$$\exists x_5 \in X\ \forall x_4 \in x_5\ \exists x_3 \in x_4\ \forall x_2 \in x_3\ \exists x_1 \in x_2 \quad x_1 \subseteq B'\Lambda\ .$$

Since II wins G_x for any $x \subseteq B'\Lambda$ this $\varphi(B'\Lambda,X)$ certainly implies that I wins G_X. (in 8 moves in fact) $\varphi(A,B)$ is a stratified wff in which B is of type 6 and A is of type 1, whence

$$\varphi(B'\Lambda,X) \longleftrightarrow \varphi((j'\pi)'B'\Lambda,(j^6{}'\pi)'X)$$

for any permutation π. But $(j^6{}'\pi)'X = X$ since X is 6-symmetric so

$$\varphi(B'\Lambda,X) \longleftrightarrow \varphi((j'\pi)'B'\Lambda,X) \text{ for all permutations } \pi.$$

Let us now suppose that I does not have a strategy to Win in 8 moves. Then $\sim\varphi(B'\Lambda,X)$, and indeed $\sim\varphi(Y,X)$ for any Y which is 1-equivalent to $B'\Lambda$. One such is $-P'B'\Lambda$, whence $\sim\varphi(-P'B'\Lambda,X)$ which is

$$\forall x_5 \in X\ \exists x_4 \in x_5\ \forall x_3 \in x_4\ \exists x_2 \in x_3\ \forall x_1 \in x_2 \quad \underline{x_1 \not\subseteq -P'B'\Lambda}$$

the underlined condition simplifies to

$$\exists x_0 \in x_1\ \ \forall x_{-1} \in x_0 \quad \Lambda \in x_{-1}$$

which is to say II Wins in 9 moves. The proofs for other finite n are similar.

$\in$-determinacy can thus have no counterexamples which are sets definable by stratified expressions.

$\in$-determinacy gets rid of Quine atoms for us. (Only one play possible in G_x if $x = \{x\}$ and that never terminates !) But there are equally pathological objects that it does not get rid of such as $x = \{x,\Lambda\}$. Such an object clearly belongs to I so it does not contradict $\in$-determinacy. Strong extensionality limits the number of such objects to 1 but does not get rid of them altogether. We shall find such an axiom in the next section where the discussion has been broadened a bit.

I am going to introduce some canonical objects, canonical in the sense that they are distinguished representatives of their kind generated in a very natural way by the theory. Whether or not they are to be set will be left open. First is

The Canonical Topology.

The pseudorank function given by $\in$-determinacy may eventually give us some constructive control over V but until we have that sort of wellfoundedness available again it is more natural to look instead from the top downwards and classify sets according to what the top few layers of them look like. For this we naturally turn to n-equivalence classes. We topologise V by taking a basis consisting of all sets of the form $[x]_n$. All neighborhoods will in fact be clopen. If we use Quine ordered pairs (so $V = V \times V$) we find that the product topology on V^2 is in fact identical with the topology on V. The fact mentioned earlier that

$$\varphi(x,y,z,\ldots) \leftrightarrow \varphi((j^{n}{}'\pi)'x,\ (j^{m}{}'\pi)'y,\ (j^{k}{}'\pi)'z,\ \ldots)$$

where n, m, k, ... are the types of x, y, z, ... can accordingly be summarised as

Fact.

Functions defined by stratified formulae are continuous. We have already seen that Strong extensionality makes the Canonical Topology T_0. The bad news is that

Fact.

The canonical topology is incompact.

In the presence of AC_2 we can find a permutation π so that π and $j'\pi$ are conjugate (see [1]). This amounts to saying $[\pi]_k = [j'\pi]_k$ for some fixed small k. Also we can show, for any n, that

$$[\delta]_n = [j'\delta]_n \rightarrow [j'\delta]_{n+1} = [j^{2}{}'\delta]_{n+1} \text{ for any permutation } \delta .$$

From this it follows that $[\pi]_k$, $[j'\pi]_{k+1}$, $[j^{2}{}'\pi]_{k+2}$, ..., $[j^{n}{}'\pi]_{k+n}$ is a nested sequence of closed sets, whose intersection must be a singleton {a}, say, if the canonical topology is compact. It then follows that a = j'a, which is to say that a is an automorphism. Any two objects that are interchanged by an automorphism must be ∞-equivalent, so strong extensionality will imply that there are no non-trivial automorphisms, contradicting compactness.

A set is <u>symmetric</u> if it is isolated in the canonical topology.

x is <u>n-symmetric</u> if $[x]_n = \{x\}$.

That is to say, x is symmetric iff it is n-symmetric for some n. The terminology "symmetric" is motivated by the fact that an n-symmetric set is fixed by lots of permutations of V, viz : all those that are j^n of something. All sets definable by stratified expressions will be symmetric. This suggest that the family of symmetric sets might be an appropriate model for any set of axioms we wish to develop. This possibility is discussed in [1] where it is shown in NF that if SYMM (the family of symmetric sets) is extensional (i.e., if x, y are distinct symmetric sets then $x \Delta y$ has a symmetric member) then it is a submodel of V elementary for stratified wffs, and that AC_2 will fail. One could motivate an axiom V = SYMM rather along the following lines : strong extensionality implies that $[x]_\infty = \{x\}$ for all x, and V = SYMM says that for each x and some n, $\{x\} = [x]_n$ already. So "V = SYMM" is a natural strengthening of strong extensionality, but its consequences are too bizarre for that to be a sufficient reason to adopt it.

A <u>permutation model</u> obtained from V and a permutation π in it, (notated V^π) is the structure obtained by keeping the same elements but rewriting ε so that $x \in y$ (in the new sense) iff $x \in \pi'y$ (in the old sense). Such models have been of great help in the devising of independence and consistency results in NF since the transition to a permutation model preserves all stratified sentences true in the original model, and all the axioms of NF are stratified. To procede further we shall need some notation. Let γ be an arbitrary permutation

$$\gamma_0 = \text{identity}. \qquad \gamma_{n+1} = (j'\gamma_n)|\gamma \quad .$$

Now we can express the following piece of folklore

$$V^{\gamma} \models \varphi(x,y,z,\ldots) \leftrightarrow V \models \varphi(\gamma_n{}'x, \gamma_k{}'y\,, \gamma_m{}'z\ \ldots)$$

where n, k, m are the types of x, y, z in φ.

In particular, $V^{\gamma} \models x \sim_n y$ iff $\gamma_n{}'x \sim_n \gamma_n{}'y$.
We shall now try to identity n-equivalence classes across permutation models. We will need an analogue of the j operation for maps $\pi : V^{\sigma} \leftrightarrow V^{\gamma}$. Calk it $m'\pi$ (a nonce notation). We have

$$(m'\pi)'x = (\pi''x) \text{ in the sense of } V^{\gamma}$$

so
$$m'\pi = \smile\gamma|j'\pi|\sigma$$
$$m^{2}{}'\pi = \smile\gamma|\smile(j'\gamma)|j^{2}{}'\pi|j'\sigma|\sigma = \smile\gamma_2|j^{2}{}'\pi|\sigma_2\,.$$

So to say that x (in V^{σ}) is n-equivalent to y (in V^{γ}) becomes

$$(\exists\pi)(y = \smile\gamma_n|(j^{n}{}'\pi)|\sigma_n{}'x)\ .$$

If we set π = identity we see that $\smile\gamma_n|\sigma_n{}'x$ is an object in V^{γ} which has the same n-equivalence class in V^{γ} as x does in V. In other words, <u>V^{σ} and V^{γ} have the same n-equivalence classes</u>. Another way of expressing this is to say that the canonical topologies in V^{σ}, V^{γ} have the same lattice of open sets and that the only difference is which nested sequences of closed sets have empty intersection. By judicious choice of τ we can arrange for V^{τ} to have, or be free of, Quine atoms. Assuming strong extensionality the (non)-existence of Quine atoms is equivalent to the following sequence of closed sets :

$$[\{\Lambda\}]_1\ , [\{\{\Lambda\}\}]_2\ ,\ [\{\{\{\Lambda\}\}\}]_3\ , \ldots$$

having nomenpty (empty) intersection. This motivates a partial order of permutations where $\sigma \leqslant \tau$ if more intersections of closed sets of the canonical topology are empty in V^{σ} than in V^{τ}. Define
$\sigma \leqslant \tau$ iff $(\exists f)(\forall x)(\forall n)(\exists\pi)(f'x = \smile\tau_n|j^{n}{}'\pi|\sigma_n{}'x)$.
Thus σ precedes τ iff we can find a function f which sends each $x \in V^{\sigma}$ to something $f'x \in V^{\tau}$ which is n-equivalent to it for each n. It is mechanical to verify that $\leqslant$ is transitive (take compositions). It is not actually antisymmetrical because $\sigma \leqslant j'\sigma \leqslant \sigma$. $\leqslant$ has an automorphism generated by $-$,

the complementation function. $-$ commutes with everything in J_1 so $j^{n}{}'-$ commutes with everything in J_{n+1}. We can use this fact to verify that $\sigma \leqslant \tau$ iff $-|\sigma \leqslant -|\tau$. Permutations can be used to give us models free of rubbish like Quine atoms. One might well feel that any creature that can be thus eradicated is probably something we are better off without. This motivates the

<u>Axiom of Minimality</u> $= \leqslant \tau$ for all permutations τ.

Minimality is the promised axiom for getting rid of things like $x = \{x,\Lambda\}$. The sweep made by minimality may be cleaner even than that, since no-one has yet proved that if V contains an infinite Von Neumann ordinal so must all its permutation models. If this is not true, then minimality would compel us to assert that all Von Neumann ordinals are finite. The reader may feel that the absence of infinite Von Neumann ordinals is unfortunate. But to do arithmetic one does not need Von Neumann ordinals any more than fingers. It does suggest, however, that if we adopt $\in$-determinacy we will find that there are no sets of infinite rank, since $(\forall\alpha)(V_\alpha \subseteq I_\alpha \cup II_\alpha)$. However that will depend on the versions of replacement and comprehension giving us, from a wellorder or length α, the Von Neumann ordinal α.

There are various ways out of this : one could weaken minimality so that it does not exclude Von Neumann ordinals. Alternately, one could adopt the point of view that Von Neumann ordinals are pathological objects, of no more mathematical interest than fingers. After some work has been done on this one will perhaps see clearly which permutations we wish to invoke minimality for.

If we consider the special case of minimality that asserts that identity $\leqslant$ complementation we infer (since $\sigma \leqslant \tau$ iff $-|\sigma \leqslant -|\tau$) that complementation $\leqslant$ identity and thus that $V \simeq V^-$. This particular case has other motivations :

Let $\hat{\varphi}$ be obtained from φ by replacing $\in$ by $\notin$ throughout and leaving "=" alone. Evidently

$$\text{LPC} \vdash \varphi \quad \text{iff} \quad \text{LPC} \vdash \hat{\varphi} \quad \text{and}$$

$$\hat{\hat{\varphi}} \text{ is } \varphi \quad .$$

(If we wish to define $\hat{}$ in a richer language notice that we negate all atomic wffs that do not contain "=". = is always inviolate. This curious circumstance merits reflection).

When φ is a statement about small sets, $\hat{\varphi}$ is a statement about big sets. This suggests that we adopt as an axiom a scheme $\varphi \leftrightarrow \hat{\varphi}$ or even something stronger. A model theoretic argument is available to show that whenever $< M,\in,= >$ and $< M,\notin,= >$ are elementarily equivalent one can find M' such that both the above are elementarily equivalent to

$$< M',\in,= > \text{ which is isomorphic to } < M',\notin,= >.$$

That is, we might as well postulate the existence of a map (possibly a proper class) σ such that $(\forall x)(\forall y)(x \in y \leftrightarrow \sigma'x \notin \sigma'y)$.
Such "antimorphisms" are discussed in [1] where it is shown that the existence of antimorphisms which are sets contradicts AC_2. That proof does not work if the antimorphism is merely a proper class.

Fact.
Any antimorphism of V is unique.

Proof.
A product of two antimorphisms is an automorphism, of which there are none, so there is a unique antimorphism, and that is of order 2.
Pending a decision on minimality we shall at least adopt one special case of it :
Axiom of Duality.
There is a unique antimorphism.

Duality and strong extensionality together get rid of another class of pathological object, the Boffa atom. x is a Boffa atom if x = B'x (That is, if $x = \{y : x \in y\}$). The reader may verify that if x is a Boffa atom and π an antimorphism then $\pi'x$ is also a Boffa atom. Also that $\pi'x$ is self-membered iff x isn't. If there are any Boffa atoms at all, there must be 4, by duality. For consider there is only one, called one. Then one $\neq \pi$'one, since one is self-membered iff π'one isn't. So there is another, call him two. Notice one $\in$ two iff two $\in$ one, since they are Boffa atoms. By by duality, if there is a pair of Boffa atoms that are members of each other there must also be another pair which are not. So there must be at least two self-membered Boffa atoms. But it is easy to see that if x and y are self-membered Boffa atoms then II Wins $G_{x=y}$: II repeatedly plays a map π where $\pi'z = z$ if z contains neither x nor y, or both. If $x \in z$ then $\pi'z = z - \{x\} \cup \{y\}$ and conversely if $y \in z$.

If there is to be a unique antimorphism we had better set about finding it. If π is an antimorphism it must satisfy the identity φ : $\pi = j'\pi|-$ (- is the complementation function).

This suggests that we devise π by approximation thus

$$\pi = \ldots\ldots |j^{n}{}'-| \ldots\ldots |j'-|- \quad .$$

The infinitary expression of the right-hand side is easily seen to satisfy the identity φ. We now note that — is of order 2 and so is $j^{n}{}'-$ for any n. So $j^{n}{}'-$, $j^{k}{}'-$ commute with one another so we can rewrite any finite approximation to the right-hand side as

$$a_n : -|j'-|j^{2}{}'-| \ldots\ldots |j^{n}{}'- \quad .$$

If we apply this permutation to some k-symmetric set, with k less than n, we can ignore the last n-k terms on the right, since they will not move anything that is k-symmetric. So if x is k-symmetric $a_n'x = a_m'x$ for any $n,m \geqslant k$, and it is this eventually-constant value of the a_n that we will take to be the value of the canonical antimorphism for argument x. It is now easy to verify that the canonical antimorphism is indeed an antimorphism <u>on the symmetric sets</u>. Any attempt to extend it to all sets meets only partial success : Let $\sigma'x$ be the canonical antimorphism, defined as above for symmetric sets, and to be $\bigcap \bigcap_{n<\omega} [a_n'x]_n$ otherwise. Of course this limit might be empty but if it isn't we argue as follows in the case x is n-symmetric and y not symmetric (the reverse case is similar and is ommitted)

$$x \in y$$

$$\leftrightarrow \quad a_n{}'x \notin a_{n+1}{}'y$$

$\leftrightarrow (j^{n}{}'\pi)'a_n{}'x \notin (j^{n+1}{}'\pi)'a_{n+1}{}'y$ for any π. But $a_n{}'x$ is n-symmetric so

$a_n{}'x \notin (j^{n+1}{}'\pi)'a_{n+1}{}'y$ for any π. In particular $a_n{}'x$ is not a member of $\sigma'y$.

In the case where neither x nor y are symmetric but $\sigma'x$, $\sigma'y$ are defined we can show $x \in y$ iff $a_n{}'x \notin a_{n+1}{}'y$ for each n but we need to find a permutation π such that $(j^{n}{}'\pi)|a_n{}'x = \sigma'x$ and $(j^{n+1}{}'\pi)|a_{n+1}{}'y = \sigma y$ simultaneously and there is no obvious reason to suppose this can be done.

The set-theoretic treatment above has been far from rigorous, and no consistency proofs are on offer. This second point should be seen as good news rather than bad, since rather than saying to us that there are no sensible set theories with a universal set, it tells us that they offer us a glimpse of a world so different that interpretations of it in terms of the old are not easy to come by. Besides, history shows that where the available mathematics is sufficiently absorbing, mathematicians are much more like to get on with developing it than worry about whether it is consistent or not. The philosophical ramifications of set theory with a universal set are simply too tempting to be ignored indefinitely.

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